Direct Nuclear Reactions

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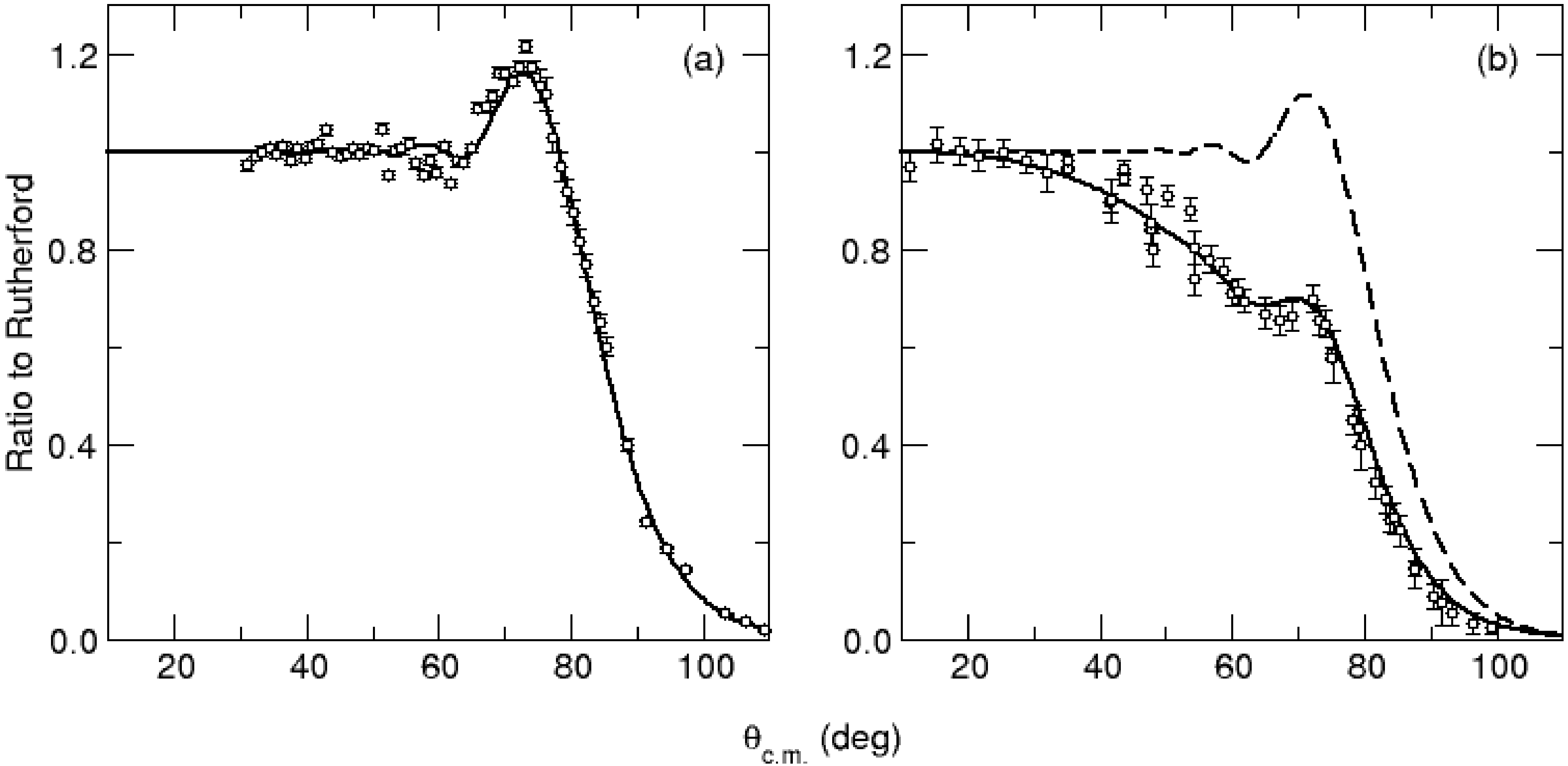
The aim of these lectures:

1) What is a direct reaction?

2) Why should we study them?

3) How do we interpret them?

^{16}O + ^{208}Pb ^{18}O + ^{184}W



A reaction is called direct if it proceeds *directly* from the initial to the final state without the formation of an intermediate compound nucleus (it takes place in one or “a few” steps)

Direct reactions are *fast*, compound reactions are *slow*

We are already in trouble: what do we really mean by “a few”, “fast” and “slow”?

The time scale is most easily quantified: we judge fast or slow relative to the motion of nucleons in the nucleus

Typically, a nucleon will take about 10^{-22} s to orbit a nucleus, and if the reaction takes place within this time scale or less, it is considered direct

Formation of a compound nucleus requires a much longer interaction time, $>> 10^{-22}$ s

In reality the distinction is not so clear cut and the two extremes shade into each other

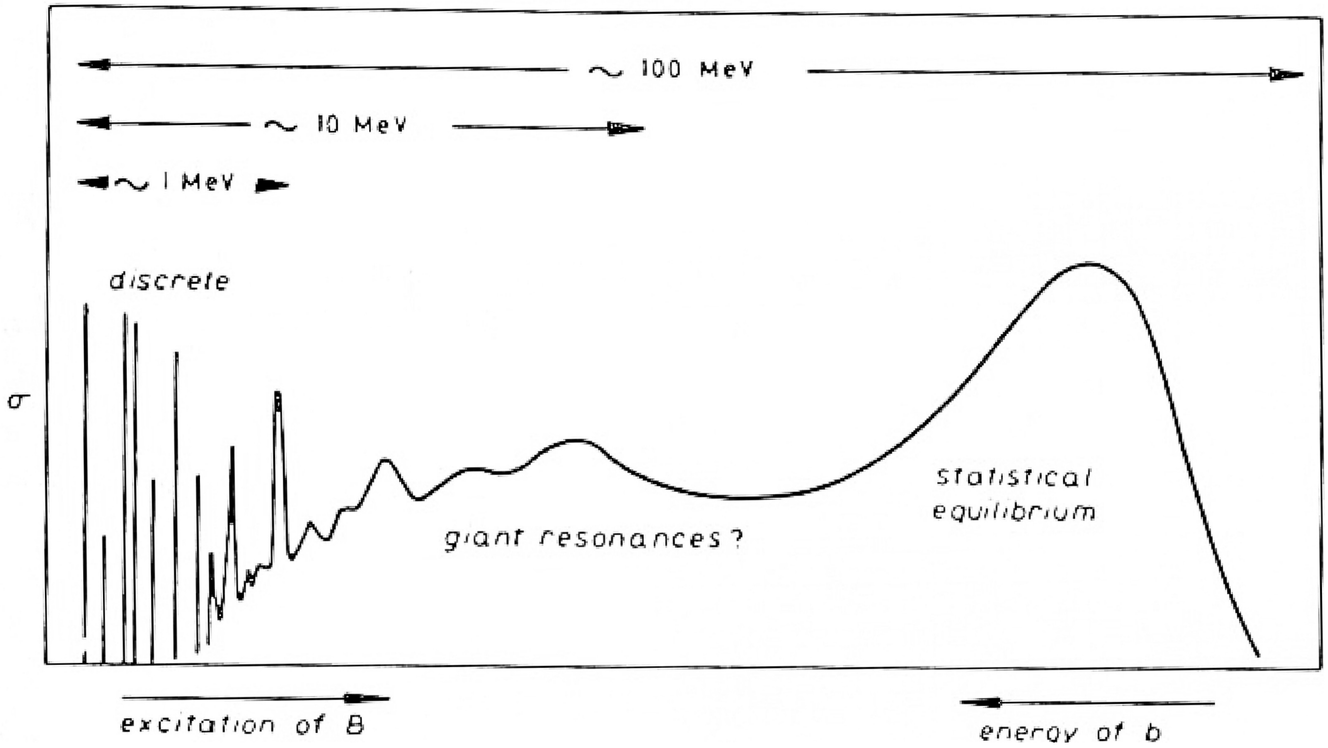
As for the number of steps involved, the distinction is even less clear-cut.

A single-step reaction is clearly direct, but …

As the projectile interacts successively it penetrates the target more deeply until eventually a compound nucleus is formed. However, the reaction may terminate after only a few interactions. Is this still direct? The distinction is often largely a question of personal taste.

Finally, direct reactions tend to be *peripheral*, i.e. they take place at the nuclear surface

Even this is not always true, since the nucleus is almost transparent to incident nucleons and direct reactions involving neutrons and protons as projectiles can occur inside the nucleus



What are the different types of (direct) reaction?

Some nomenclature:

We write a reaction thus:

$$\mathrm{A} + \mathrm{a} \rightarrow \mathrm{B} + \mathrm{b} + Q$$

A is the target, a the projectile, B the residual, b the ejectile and Q is the “Q value”.

This is written more concisely as:

$$\mathrm{A(a,b)B}$$

Each possible combination of particles is referred to as a *partition*. Further, within a partition we may specify the state of excitation of each nucleus. Such a pair of nuclei each in a specific state is known as a *channel*. Each partition can in principle consist of many channels.

Thus the combination A + a is known as the *entrance channel* (both nuclei are in their ground states). The various possible sets of products B and b in their specific excitation states constitute the *exit channels*.

The Q value is simply the energy released during the reaction. It is most simply defined as:

$$Q = E_f - E_i$$

If the Q value is negative the reaction is termed *endothermic*, i.e. the final total kinetic energy is less than the initial.

If the Q value is positive we have an *exothermic* reaction, and binding energy (or rest mass) is released during the reaction

For negative Q, there is a threshold energy since E_f must be > 0, $E_i = E_{th} = -Q$

Back to the various classes of (direct) reaction …

1) Elastic scattering. The simplest “reaction”. The internal states of a and A are unchanged and $Q = 0$. Written as: A(a,a)A

2) Inelastic scattering. Usually, this term is applied to a reaction where A is left in an excited state, i.e. B = A* and therefore $Q = -E_x$. a is then emitted with reduced energy and the reaction is written as: A(a,a′)A*.

For complex projectiles a may be excited instead of the target or both a and A may be left in excited states (mutual excitation).

3) Transfer reactions. Here B ≠ A and b ≠ a, there has been a rearrangement (transfer) of nucleons between target and projectile.

Two sub-groups, conventionally defined in terms of the projectile:

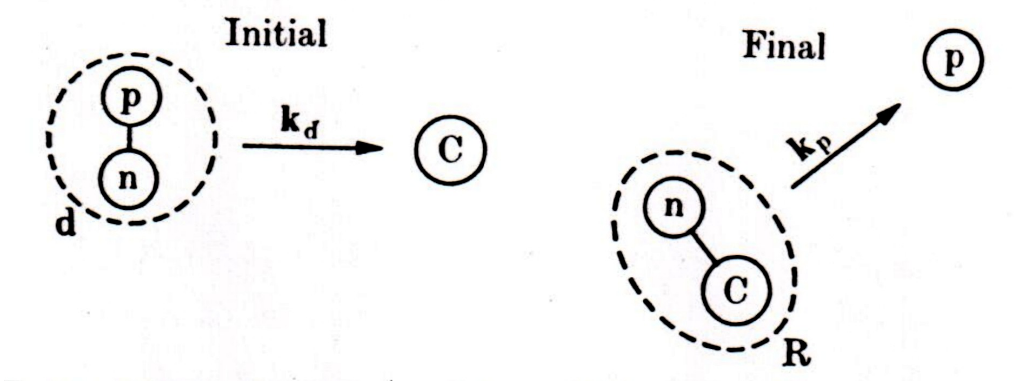
Stripping – a nucleon or nucleons is transferred from the projectile to the target

Pickup – a nucleon or nucleons is transferred from the target to the projectile

Examples:

Stripping: ^{208}Pb(d,p)^{209}Pb

Pickup: ^{208}Pb(p,d)^{207}Pb



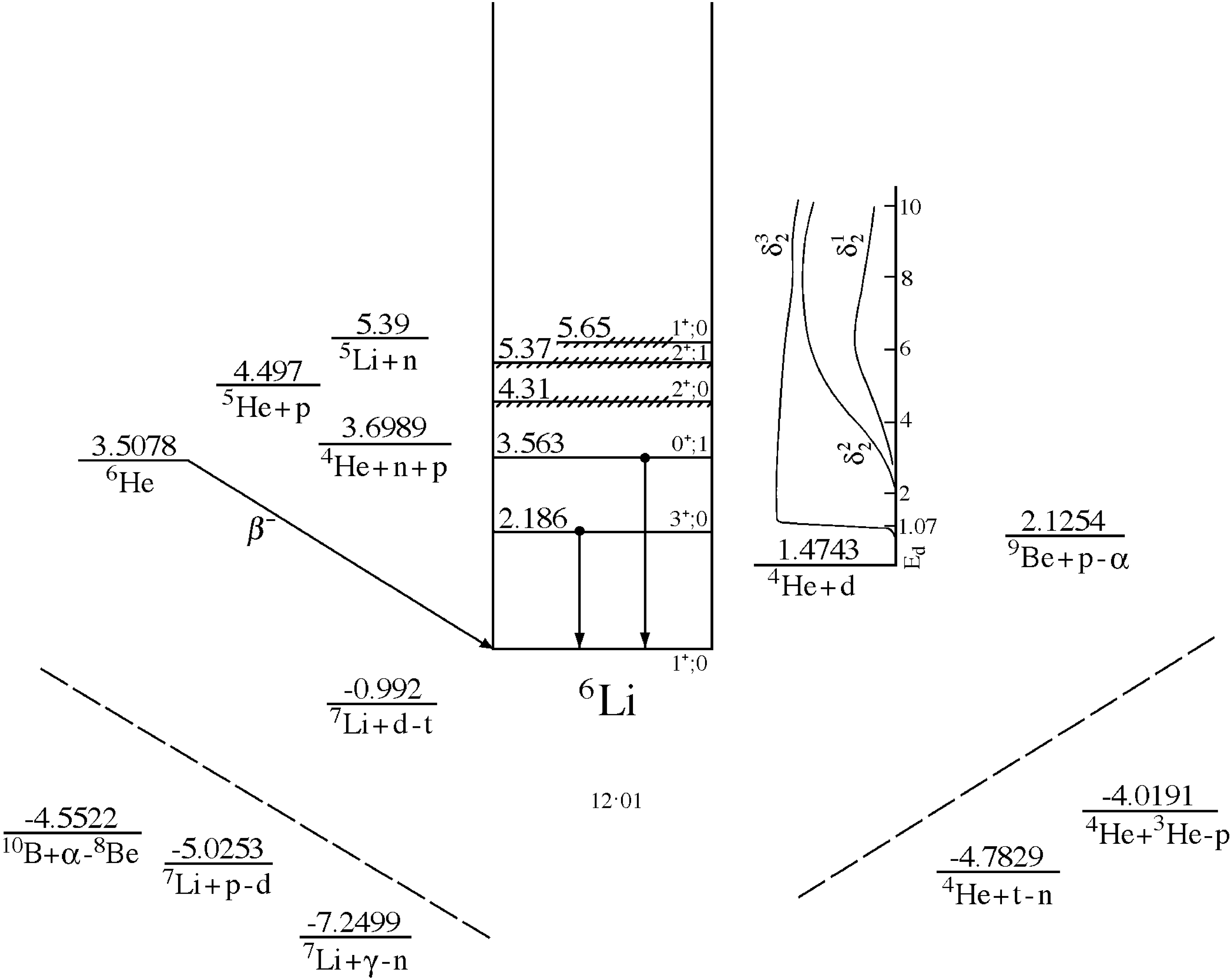
4) Breakup. This is no longer a simple two-body process but (usually) a three-body one:

$$A(a,a^* \to b + c)A$$

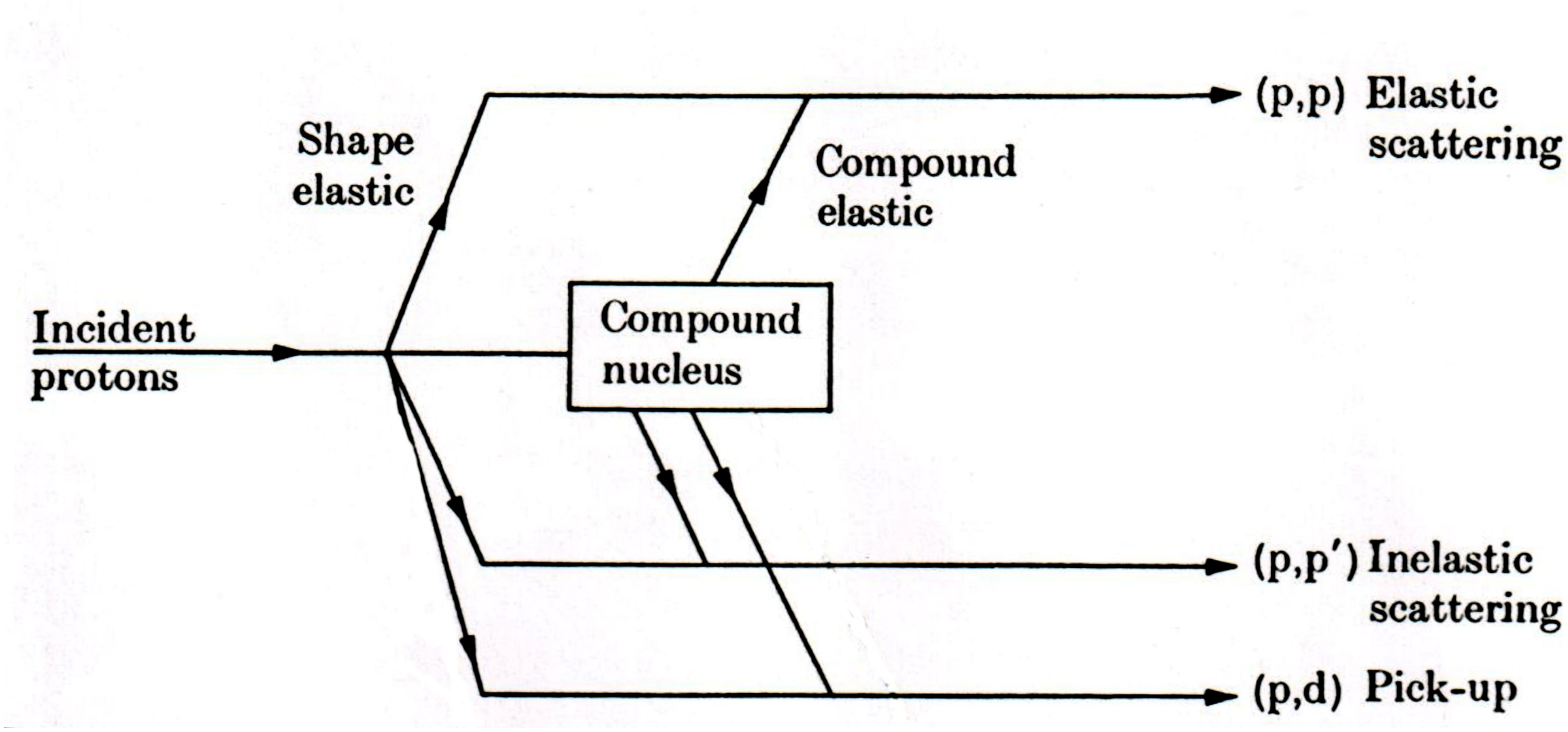
i.e. a is excited to a “state” (either a resonance or the continuum) above the emission threshold for particle c (a is considered to consist of particle c bound to core b)

Example: $^6\mathrm{Li}^* \to \alpha + d$

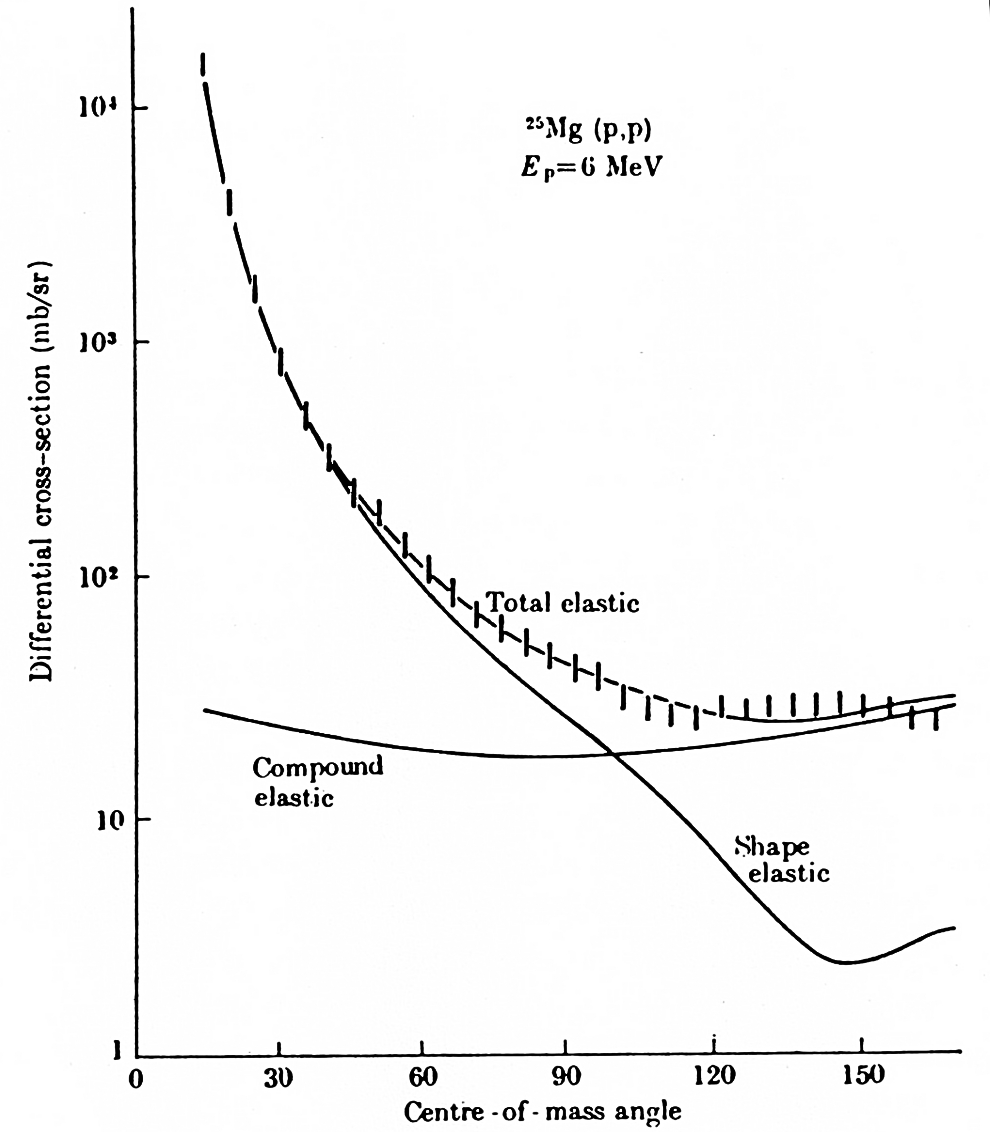
Four-body (or more) breakup modes can also occur, e.g. $^6\mathrm{He} \to \alpha + n + n$


5.39
^{5}Li+n
4.497
^{5}He+p
3.6989
^{4}He+n+p
3.5078
^{6}He
β⁻
5.65
1⁺;0
5.37
2⁺;1
4.31
2⁺;0
3.563
0⁺;1
2.186
3⁺;0
1⁺;0
^{6}Li
δ$_2^3$
δ$_2^1$
δ$_2^2$
10
8
6
4
2
1.07
E$_d$
1.4743
^{4}He+d
2.1254
^{9}Be+p-α
-0.992
^{7}Li+d-t
12·01
-4.5522
^{10}B+α-^{8}Be
-5.0253
^{7}Li+p-d
-7.2499
^{7}Li+γ-n
-4.0191
^{4}He+^{3}He-p
-4.7829
^{4}He+t-n

These are the direct reactions that we shall discuss in these lectures. However, for light ions (nucleons in particular) even these reactions can have a compound component:



A particular case is so-called compound elastic scattering. This is only a problem for low-energy nucleon elastic scattering:



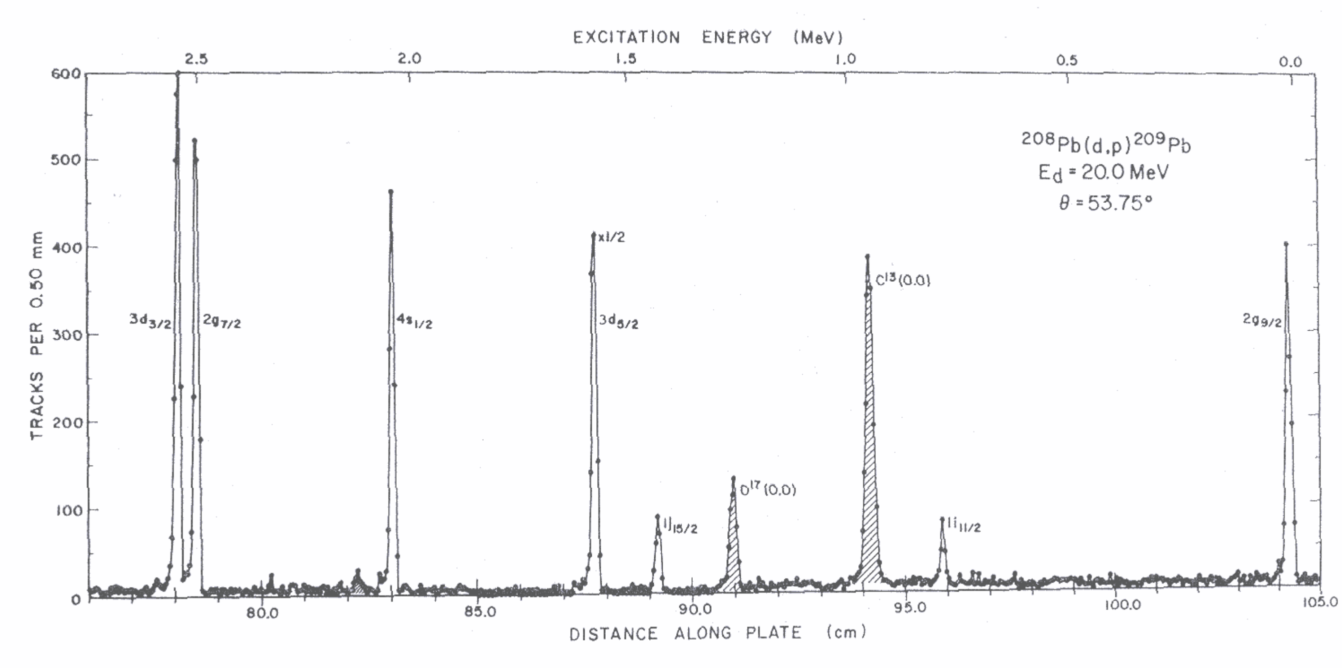
Compound elastic scattering occurs when, after formation of the compound nucleus in an excited state, a proton (in this case) is emitted with the same energy as the incident proton. As the incident energy increases this process rapidly becomes less likely, so we only have to consider it at low energies. Fortunately, compound and shape elastic (the direct scattering process) do not interfere and can be analysed separately.

Finally, if the projectile has exactly the right energy *resonant* behaviour can occur (usually only in elastic or inelastic scattering). This is not confined to nucleons or light ions, it can also occur for the lighter heavy ions (a heavy ion is by convention anything heavier than an α particle). However, it usually only occurs at low incident energies and we shall not consider it further here.

Having defined what we mean by a direct reaction and described the various types that we are interested in, a word about what we actually *measure*

What are the observables?

1) A spectrum: we measure the energy of the outgoing particle(s) as a function of the scattering angle:



2) We can count the number of outgoing particles (of a particular type) either in total or as a function of angle. This is usually reduced to a “standardised” number, the *cross section*

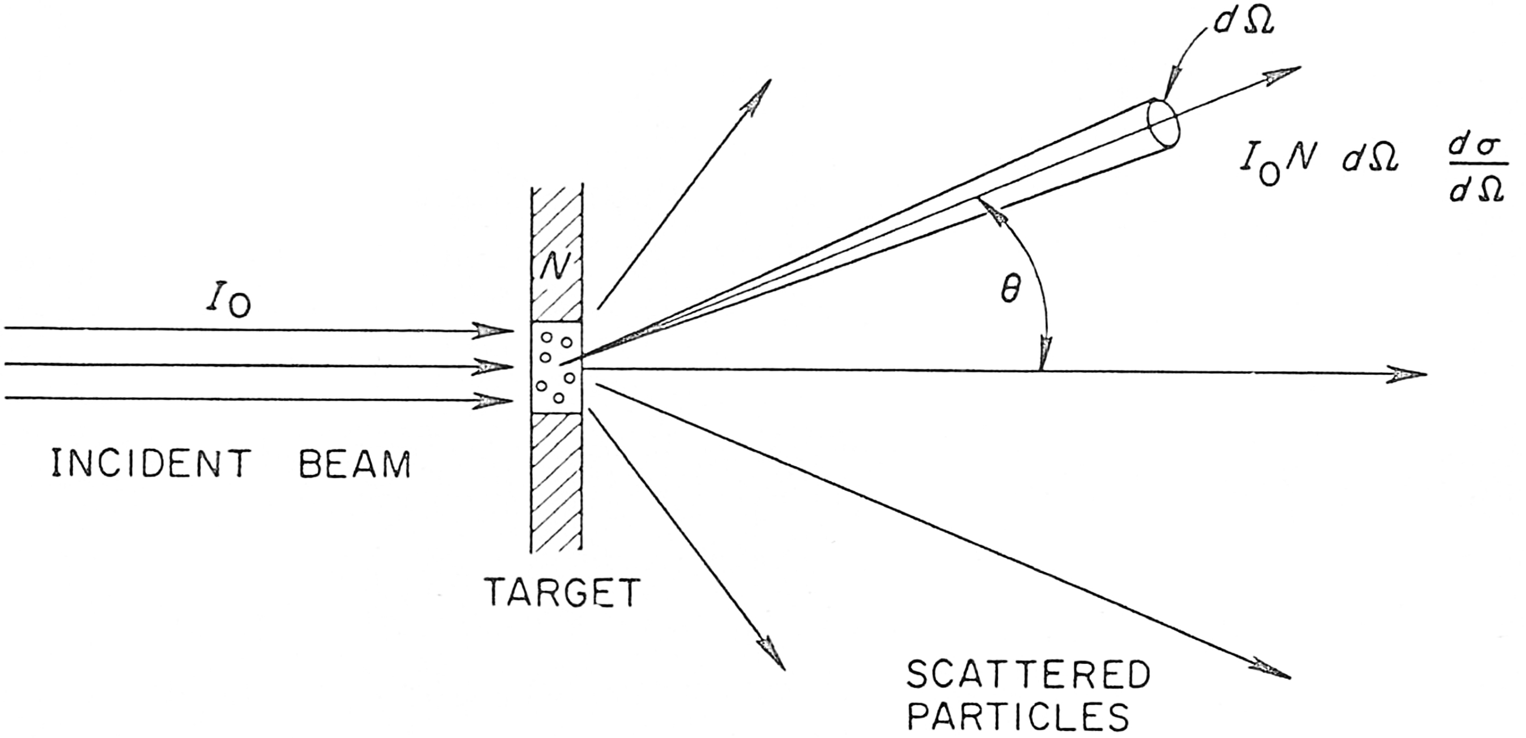
Remember that any other “experimental” quantities are in fact derived from these observables using models

How is cross section defined? Although it has units of area (usually measured in barns — symbol b, $1\text{b} = 10^{-28}\ \text{m}^2$ — or sub-multiples thereof in nuclear physics) it is really a measure of the probability that a particular reaction will occur.

Returning to our typical reaction A(a,b)B. If we have I_0 particles of type a per unit area incident on a target containing N particles of type A then the number of particles b that we detect is obviously proportional to I_0 and N, the constant of proportionality being the cross section, σ:

$$\sigma = \frac{\text{number of particles b detected}}{I_0 \text{ x } N}$$

If we count the number of particles b in the solid angle element dΩ in the direction (θ,φ) with respect to the beam direction it is proportional to dΩ as well as I_0 and N and the constant of proportionality is the differential cross section dσ/dΩ (usually measured in units of mb/sr):



In fact, unless the beam is polarised dσ/dΩ does not depend on the azimuthal angle φ and is sometimes written: σ(θ)

The two kinds of cross section are clearly related:

$$\begin{aligned}\sigma &= \int_0^{4\pi} (\mathrm{d}\sigma/\mathrm{d}\Omega)\mathrm{d}\Omega \\ &= \int_0^{\pi} \sin\theta \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi(\mathrm{d}\sigma/\mathrm{d}\Omega)\end{aligned}$$

If dσ/dΩ is independent of φ then:

$$\sigma = 2\pi \int_0^{\pi} (\mathrm{d}\sigma/\mathrm{d}\Omega) \sin\theta \mathrm{d}\theta$$

In direct reaction work we usually measure the angular distribution of the differential cross section, since this contains most of the information (as we shall see later on).

However, sometimes the cross section σ is measured. If we measure σ for each type of particle emitted in a non-elastic process this is called the *reaction cross section*. If we then add σ_{el} to this number we obtain the *total cross section*, a measure of the probability that *something* will occur during the collision

The reaction cross section, σ_R, is the most useful of these quantities and we shall meet it again later.

Finally, a word about books:

Introduction to Nuclear Reactions, G. R. Satchler two editions, 1980 and 1991 (paperback). Still the best introduction to the subject but long out of print and difficult to find.

Direct Nuclear Reactions, G. R. Satchler, OUP 1983. Long out of print and very difficult to find. Still a very good monograph on the subject.

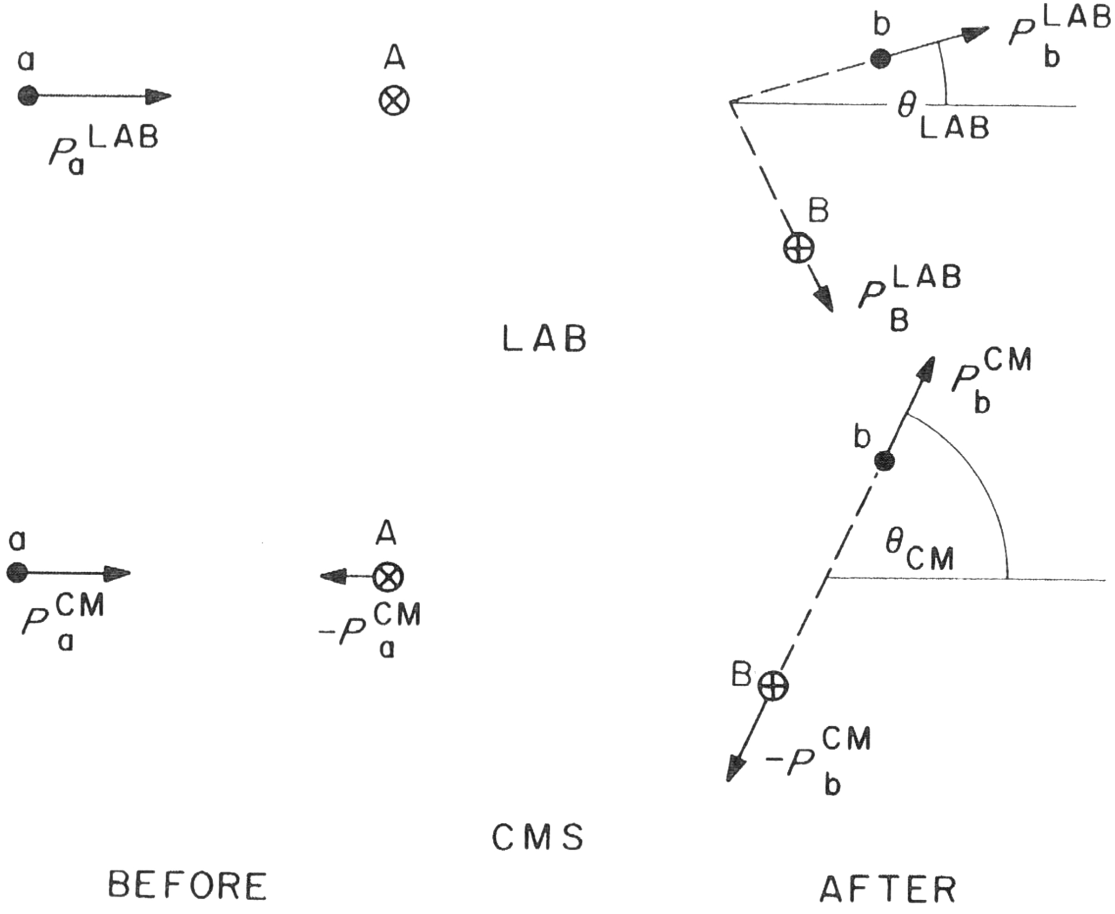
Nuclear Reactions and Nuclear Structure, P. E. Hodgson, OUP 1971. Also long out of print and difficult to find. A good monograph on light-ion induced reactions and their analysis.

Nuclear Reactions for Astrophysics: Principles, Calculation and Applications of Low-Energy Reactions, I. J. Thompson and F. M. Nunes, CUP 2009. Perhaps the best of the in-print books on the subject. Linked to practical use of the direct reaction code FRESCO.

Lecture 2: Kinematics

“Kinematics” covers a multitude of sins. In this lecture we shall define some more quantities that will be useful later and look at the consequences of some conservation laws.

Experiments are (naturally enough …) performed in the laboratory reference frame. However, we usually transform to the centre of mass (CoM) frame before we attempt to analyse the data:



Why do we do this?

It makes life easier! One conserved quantity in a nuclear collision is the *total* momentum. We thus chose a reference system where the total momentum is 0 – the centre of mass system.

Here, the centre of mass of the projectile-target system is at rest and the projectile and target approach each other with equal and opposite momenta: $P'_a = -P'_A$. Since total momentum is conserved, $P'_b = -P'_B$ if there are only two products.

We mentioned briefly in the first lecture the *Q* value as the energy released in a reaction. Since the total energy is conserved, $Q = E_i - E_f$. This may also be written in terms of the rest masses or the binding energies of the particles involved:

$$m_A + m_a = m_B + m_b + Q/c^2$$

$$B_A + B_a = B_B + B_b - Q$$

If *Q* is negative, e.g. inelastic excitation of ^{16}O 3^- (6.13 MeV) by protons, then we require a minimum kinetic energy of the proton of 6.13 MeV *in the centre of mass frame* for the reaction to proceed

The kinetic energy of the motion of the centre of mass of the system is conserved and is therefore not available for producing nuclear excitations: advantage of CoM system is that it is zero (CoM at rest, by definition) so we do not need to worry about it!

Other conserved quantities are total charge, number of neutrons and protons (in the reactions we will consider; it is not always so), the *parity* and the *total angular momentum.* Any change in the total (vector sum) of the intrinsic spins of the nuclei must be compensated for by a change in the orbital angular momentum of their relative motion. Likewise for the product of their intrinsic parities.

Although not really a “kinematical” quantity we shall briefly discuss here the *Coulomb barrier*.

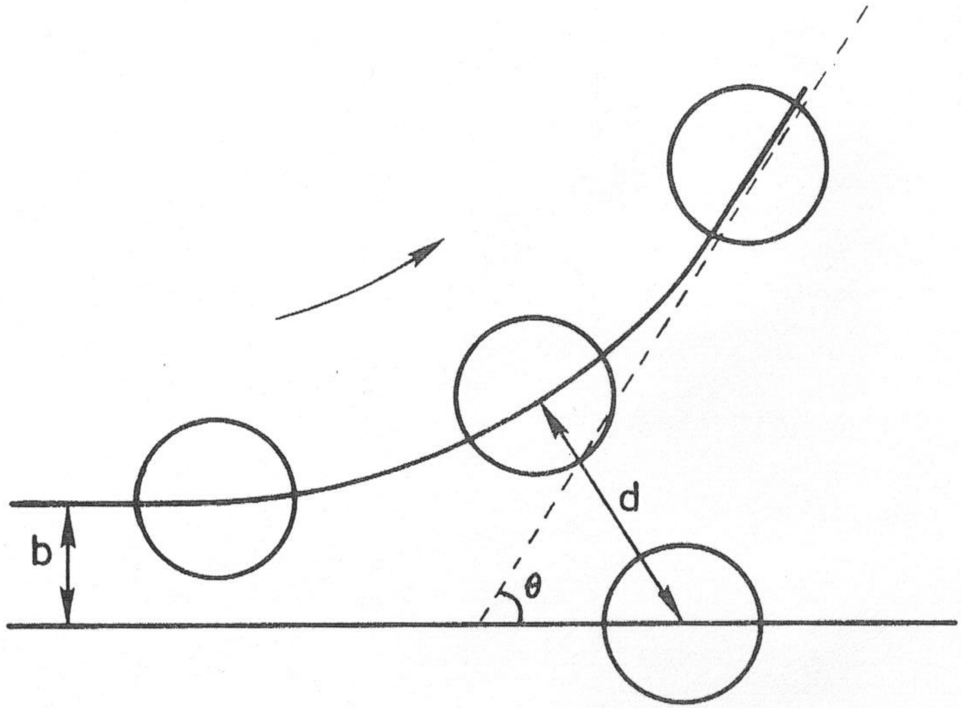
There are almost as many (slightly) differing definitions of this quantity as there are nuclear physicists!

A good working definition is the value of the Coulomb potential for two touching charged spheres with the same charges as the colliding nuclei:

$$E_B = \frac{Z_1 Z_2 e^2}{R_1 + R_2}$$

with $R = 1.16\ A^{1/3} + 1.2$ fm

To define some other useful quantities we consider a classical orbit (most useful for heavy ion projectiles)



b is the *impact parameter*, d the *distance of closest approach* and θ the CoM scattering angle

For scattering from a purely Coulomb potential (i.e. as in Rutherford scattering) we have:

$$b = a \cos \tfrac{1}{2} \theta$$

and

$$d = a\,(1 + \operatorname{cosec} \tfrac{1}{2} \theta)$$

where

$$a = Z_1 Z_2 e^2/mv^2$$

or, $a = d_0/2$ where $d_0 = d$ for a head-on collision ($b=0$)

For the general case of a particle scattered by a central potential V(r) (which will be the sum of a nuclear and a Coulomb potential) the relation between θ and b — the classical deflection function — becomes:

$$\theta(b) = \pi - 2b \int\limits_{r_{\min}}^{\infty} \frac{1}{r^2} \left\{ 1 - \frac{V(r)}{E} - \frac{b^2}{r^2} \right\}^{-1/2} \mathrm{d}r$$

r_{min} is the distance of closest approach, i.e. the value of r that reduces the expression:

$$1 - \frac{V(r)}{E} - \frac{b^2}{r^2}$$

to zero.

Some other useful quantities for Coulomb orbits are:

$$n = \frac{Z_1 Z_2 e^2}{\hbar v} \approx \frac{Z_1 Z_2}{6.35} \left(\frac{m(\mathrm{u})}{E(\mathrm{MeV})} \right)^{1/2}$$

the *Sommerfeld parameter*, n.

$k = 1/ƛ = (2mE/\hbar^2)^{1/2} \approx 0.2187[m(u)E(MeV)]^{1/2}\ fm^{-1}$

the *wave number*, k (note that momentum = ħk).

$L = n \cot(\tfrac{1}{2}\theta) = kb$

the angular momentum of the projectile about the target (in units of ħ).

For heavy ions the concept of the *critical* or *grazing angle,* θ_c, is useful (as we shall see later in these lectures).

θ_c is defined as the scattering angle for the orbit where the nuclear surfaces just touch. Since the nuclear forces fall off very rapidly beyond the nuclear surface we may use the relations for Coulomb orbits with little loss of precision (for heavy ions)

If R is the sum of the nuclear radii, then we have:

$$R = a\,(1 + \operatorname{cosec} \tfrac{1}{2}\theta_c)$$

If we substitute for a, and recall that $E = \tfrac{1}{2} mv^2$:

$$\begin{aligned}\operatorname{cosec}\left(\frac{1}{2}\theta_c\right) &= \frac{R}{a} - 1 \\ &= \frac{2ER}{Z_1 Z_2 \mathrm{e}^2} - 1 \\ &= \frac{2r_0\left(A_1^{1/3} + A_2^{1/3}\right)E}{Z_1 Z_2 \mathrm{e}^2} - 1\end{aligned}$$

where we have substituted $R = r_0 (A_1^{1/3} + A_2^{1/3})$.

One could of course use more sophisticated expressions for R. We shall see later that one can also extract “empirical” values for θ_c from data and that θ_c is a useful parameter in heavy-ion reactions.

All these definitions are based on *classical* concepts. This is a reasonable approximation provided that:

$$n >> 1$$

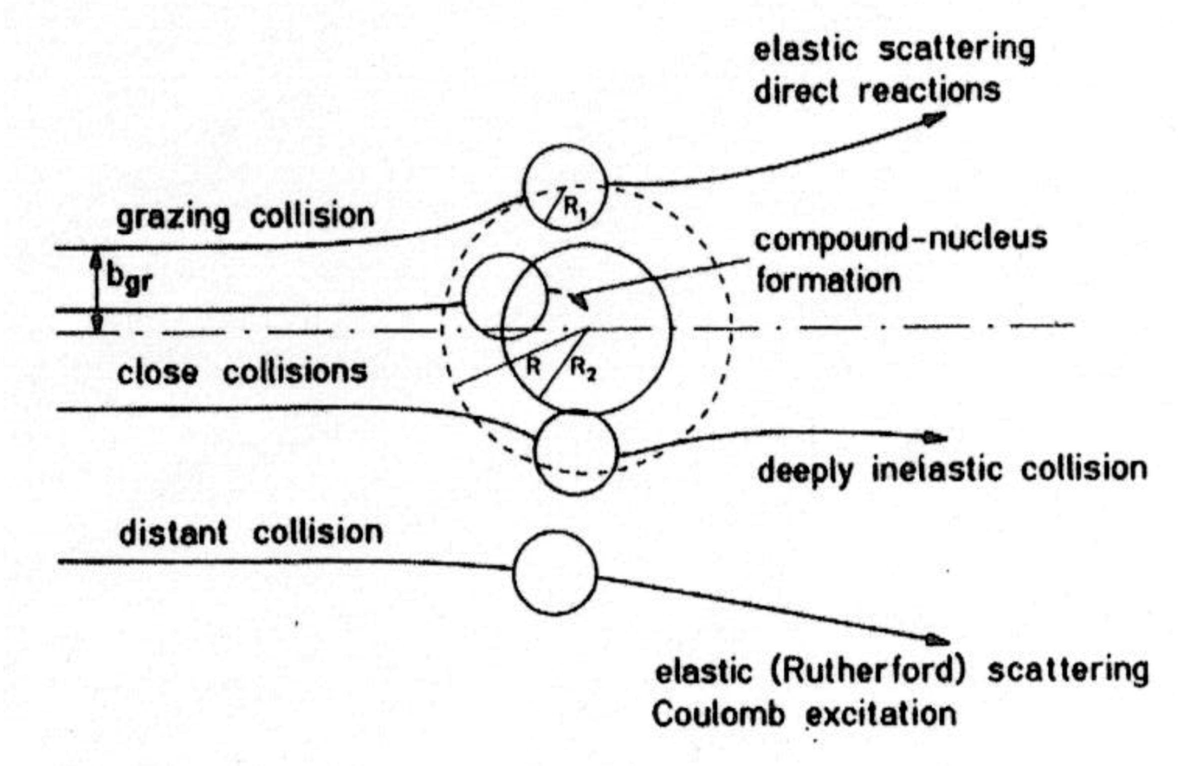
This condition is often satisfied for heavy ions (particularly at low energies) because they are massive and carry large charges.

We can further refine these ideas to include some quantum concepts and produce semi-classical theories. Some of these concepts are nevertheless useful even for light-ion induced reactions …

For example, the impact parameter, b. Recall that:

$$b = a \cos \tfrac{1}{2} \theta$$

Therefore, small impact parameters (head-on or near head-on collisions) correspond to large scattering angles:



Lecture 2: Scattering theory

The full quantum mechanical treatment of scattering. The classical and semi-classical treatments have their uses (as we have just seen) but we need to cover the essentials of the quantum mechanical theory here since it underlies the analysis methods we shall describe later.

Those interested in going further are referred to the books by Satchler (Direct Nuclear Reactions), Austern (Direct Nuclear Reaction Theories, Wiley 1970) and Glendenning (Direct Nuclear Reactions, Academic Press 1983 or World Scientific 2004).

Our goal in the theoretical analysis of direct reaction data is to calculate a differential cross section. To do this we need to solve the appropriate Schrödinger equation. For a single particle of mass *m* (and no internal degrees of freedom) scattered by a potential *V*(**r**) it is:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\chi(\mathbf{r}) = E\chi(\mathbf{r})$$

Outside the region where *V* acts *E* is the kinetic energy
We may rewrite this in the simplified form:

$$H\chi(\mathbf{r}) = E\chi(\mathbf{r})$$

where *H* is the *Hamiltonian* for the system

We may also re-write the equation in the following more convenient form:

$$\left[\nabla^2 - U(\mathbf{r}) + k^2\right]\chi(\mathbf{r}) = 0$$

where $U = 2mV/\hbar^2$. For $V = 0$ the solution is a plane wave $\chi = \exp(i\mathbf{k}.\mathbf{r})$

This equation still holds for the interaction of two complex nuclei a and A if we interpret **r** as $\mathbf{r}_\alpha$, the distance between their centres of mass and replace m with the reduced mass $\mu_\alpha = m_a m_A/(m_a + m_A)$, *provided V* cannot change the internal states of a and A (i.e. only elastic scattering can occur)

This is a somewhat artificial situation of course. Usually *V* can not only excite either or both of a and A but can also lead to a rearrangement reaction to give two different nuclei b and B.

We may define *internal* Hamiltonians for a and A:

$$H_\mathrm{a}\,\psi_\mathrm{a} = \varepsilon_\mathrm{a}\,\psi_\mathrm{a},\quad H_\mathrm{A}\,\psi_\mathrm{A} = \varepsilon_\mathrm{A}\,\psi_\mathrm{A}$$

The total Hamiltonian then becomes:

$$H = H_\mathrm{a} + H_\mathrm{A} - \frac{\hbar^2}{2\mu_\alpha}\nabla_\alpha^2 + V_\alpha$$

or, $H\,\Psi = E\,\Psi$

The ψ_a and ψ_A form complete sets, so we may expand the total wave function Ψ as follows:

$$\Psi = \sum_{\mathrm{a'A'}} \chi_{\mathrm{a'A'}}(\mathbf{r}_\alpha)\psi_{\mathrm{a'A'}}$$

Substituting back into the Schrödinger equation, multiplying from the left by $\psi_a{}^*\psi_A{}^*$ and integrating over the internal coordinates we get (putting it into the form with $U(\mathbf{r}_\alpha)$ and k):

$$\left[\nabla_\alpha^2 - U_{\mathrm{aA,aA}}(\mathbf{r}_\alpha) + k_{\mathrm{aA}}^2\right]\chi_{\mathrm{aA}}(\mathbf{r}_\alpha) = \sum_{\substack{\mathrm{a'}\neq\mathrm{a}\\ \mathrm{A'}\neq\mathrm{A}}} \chi_{\mathrm{a'A'}}(\mathbf{r}_\alpha)U_{\mathrm{aA,a'A'}}(\mathbf{r}_\alpha)$$

where: $k_{aA} = 2\mu_\alpha (E - \varepsilon_a - \varepsilon_A)/\hbar^2$

We have introduced here the matrix elements of the interaction potential V_α:

$$\begin{aligned} U_{\mathrm{aA,a'A'}}(\mathbf{r}_\alpha) &= \frac{2\mu_\alpha}{\hbar^2} \iint \psi^*_{\mathrm{a}}(\tau_{\mathrm{a}})\psi^*_{\mathrm{A}}(\tau_{\mathrm{A}})\; V_\alpha \; \psi_{\mathrm{a'}}(\tau_{\mathrm{a}})\psi_{\mathrm{A'}}(\tau_{\mathrm{A}}) \; \mathrm{d}\tau_{\mathrm{a}} \, \mathrm{d}\tau_{\mathrm{A}} \\ &\equiv \frac{2\mu_\alpha}{\hbar^2} \langle \mathrm{aA} \mid V_\alpha \mid \mathrm{a'A'} \rangle \end{aligned}$$

The integrations are only over the *internal* coordinates τ_a and τ_A.

The *diagonal* element is on the LHS and the *off-diagonal* elements on the RHS in our example. The LHS by itself describes elastic scattering only; the RHS describes the inelastic scattering and its effect on the elastic scattering.

Similar equations exist for all pairs a′A′ obtained by the same procedure but with $\psi_{a'}{}^{*}\psi_{A'}{}^{*}$. There is thus an infinite set of *coupled equations*, one for each pair a,A (each *channel*).

Since we obviously cannot solve an infinite set of such equations, we truncate by including only those channels that we know (or guess) to be strongly coupled. The rest are either neglected or we represent their effect by a complex *optical potential*.

It is at this point that models enter and we have the *coupled channels (CC)* method.

This is fine for inelastic scattering, and it could also be used for rearrangement (transfer) reactions.

However, in practice this is not feasible since it would require a very large number of states a′,A′ so other techniques are used for these reactions.

Finally (in this lecture!) how do we calculate the cross section once we have the wave function? By applying appropriate boundary conditions we can define the *scattering amplitude*:

$$f(\theta, \phi) = -\frac{1}{4\pi} \int \exp(-i\mathbf{k}' \cdot \mathbf{r}') \, U(\mathbf{r}') \, \chi(\mathbf{k}, \mathbf{r}') \, \mathrm{d}\mathbf{r}'$$

We then define the differential cross section as:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \mid f(\theta,\phi) \mid^2$$

It may look as if this doesn't help much, since we still do not know what $\chi(\mathbf{r}')$ is in $f(\theta,\varphi)$. However, it does provide a very good starting point for approximations, as we shall see next time.

Lecture 3: Scattering theory continued

Last time we saw that we may calculate the differential cross section from the scattering amplitude, which may be formulated thus:

$$f(\theta, \phi) = -\frac{1}{4\pi} \int \exp(-i\mathbf{k}' \cdot \mathbf{r}')\, U(\mathbf{r}')\, \chi(\mathbf{k}, \mathbf{r}')\, \mathrm{d}\mathbf{r}'$$

where **k**′ is the momentum of the outgoing particle.

As we noted last time, this is only a formal solution of the problem since we still don't know χ(**k**,**r**′)

However, we are now in a position to produce approximate solutions to the scattering problem.

In the simplest case, if potential *V* is weak we may approximate χ(**k**,**r**′) by the incident plane wave:

$$f_{\mathrm{BA}}(\theta,\phi) = -\frac{1}{4\pi}\int \exp(-i\mathbf{k}'\cdot\mathbf{r}')\,U(\mathbf{r}')\;\exp(i\mathbf{k}\cdot\mathbf{r}')\,\mathrm{d}\mathbf{r}'$$

$$= -\frac{1}{4\pi}\int \exp(i\mathbf{q}\cdot\mathbf{r}')\,U(\mathbf{r}')\,\mathrm{d}\mathbf{r}'$$

where **q** = **k** − **k**′ is the change in momentum of the scattered particle (ends up as recoil of the target)

This is the *plane wave Born approximation* (also known as *Butler theory*). No longer used, it has been replaced by the somewhat more sophisticated *distorted wave Born approximation (DWBA).*

This time, to obtain an approximation for $\chi(\mathbf{k},\mathbf{r}')$ we assume that U may be written $U = U_1 + U_2$ and that we can obtain the scattering solution for U_1:

$$\left[\nabla^2 + k^2 - U_1(\mathbf{r})\right] \chi_1(\mathbf{k},\mathbf{r}) = 0$$

We consider two types of solution, $\chi_1^{(+)}(\mathbf{k},\mathbf{r})$ and $\chi_1^{(-)}(\mathbf{k},\mathbf{r})$, consisting of plane waves plus *outgoing* and *ingoing* scattered waves respectively.

We may then find a solution of the full ($U = U_1 + U_2$) problem in terms of these $\chi_1^{(\pm)}$:

$$f(\theta,\phi) = f_1(\theta,\phi) - \frac{1}{4\pi}\int \chi_1^{(-)}(\mathbf{k}',\mathbf{r}')^* \, U_2(\mathbf{r}') \, \chi(\mathbf{k},\mathbf{r}') \, \mathrm{d}\mathbf{r}'$$

This is still only a formal (exact) solution. If we now replace χ with χ_1 (the solution for U_1) we get:

$$f_{\mathrm{DWBA}}(\theta,\phi) = f_1(\theta,\phi) - \frac{1}{4\pi}\int \chi_1^{(-)}(\mathbf{k}',\mathbf{r}')^* \, U_2(\mathbf{r}') \, \chi_1^{(+)}(\mathbf{k},\mathbf{r}') \, \mathrm{d}\mathbf{r}'$$

This will be a good approximation when U_2 is weak compared to U_1

This approximation may be generalised to inelastic and transfer reactions. We choose U_1 (and thus f_1) to describe the *elastic scattering* and U_2 is then the interaction inducing the non-elastic transition. *Thus DWBA is valid if elastic scattering dominates and other events can be treated as perturbations.*

As an example, the DWBA transition amplitude for the reaction A(a,b)B has the form:

$$f_{\mathrm{DWBA}}(\theta,\phi) = -\frac{1}{4\pi}\int \chi_{\beta}^{(-)}(\mathbf{k}_\beta,\mathbf{r}_\beta)^* \ \langle \mathrm{b},\mathrm{B} \mid U_2 \mid \mathrm{a},\mathrm{A}\rangle \ \chi_{\alpha}^{(+)}(\mathbf{k}_\alpha,\mathbf{r}_\alpha)\, \mathrm{d}\mathbf{r}_\alpha \mathrm{d}\mathbf{r}_\beta$$

Where we have generalised χ_1 to χ_α and χ_β, the solutions for elastic scattering in entrance and exit channels from potentials U_α and U_β

Finally, a few words on partial wave expansion. The relative angular momentum of two colliding particles, ℓħ, is quantised in units of ħ. Since nuclear forces are short ranged and nuclei have reasonably sharp edges only particles with angular momentum less than some maximum value interact with the target. This can be quite small (for protons, as little as ℓ = 10 – 15).

If we have a central potential (usual in nuclear reactions) angular momentum is conserved and we may write:

$$\chi_{\ell m}(\mathbf{r}) = u_\ell(\mathbf{r}) Y_\ell^m(\theta, \phi)$$

The Schrödinger equation then separates into radial and angular equations. We may most easily write the radial equation in terms of $w_\ell(r) = r\, u_\ell(r)$:

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2 w_\ell}{\mathrm{d}r^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right] w_\ell = E w_\ell$$

This enables us not only to truncate our partial wave series (although in some heavy ion reactions we may still need several hundred) but it also reduces the three dimensional Schrödinger equation to a series of one dimensional radial equations.

Lecture 3: elastic scattering and the optical model

We now begin the main subject of these lectures: how do we interpret direct reactions and why should we study them?

We start with the most fundamental process, elastic scattering. It is always present, so any direct reaction theory must take account of it in some way. Here we consider the simplest theory of elastic scattering, the *optical model*, which treats elastic scattering alone.

The optical model replaces the full scattering problem with that for scattering by a (complex) potential, the simplest case we considered in our brief look at scattering theory.

The *optical potential* must be complex to account for absorption into other reaction channels that we do not treat explicitly.

It is possible formally to construct an optical potential of this type from the full problem in scattering theory (usually referred to as "Feshbach Theory"). However, in practice it is virtually impossible to calculate such a potential which is in any case non-local and L-dependent.

Instead, we take the practical approach and consider an empirical *optical model potential* which is local and L-independent.

Our OMP will be of the form:

$$U(r) = V(r) + \mathrm{i}\, W(r) + V_C(r)$$

where $V_C(r)$ is the Coulomb potential for a uniformly charged sphere of radius R_C:

$$\begin{aligned} V_C(r) &= \frac{\mathrm{Z_I Z_T e^2}}{2R_C}\left(3 - r^2/R_C^2\right) && \text{for } \ r < R_C \\ &= \frac{\mathrm{Z_I Z_T e^2}}{r} && \text{for } \ r > R_C \end{aligned}$$

R_c is typically 1.3 x $A_T^{1/3}$ or 1.3 x $(A_I^{1/3} + A_T^{1/3})$

How do we choose the nuclear potential? Nuclear forces are short-ranged, therefore we expect them to drop off sharply outside the nucleus — square well? Too abrupt; a well with rounded edges?

Fermi distribution, known in the direct reaction world as the *Woods-Saxon distribution:*

$$f(r, R, a) = \frac{1}{1 + \exp\left(\frac{r-R}{a}\right)}$$

so that the nuclear part of the OMP becomes:

$$U_{\mathrm{N}}(r) = -(V + iW) f(r, R, a)$$

In this form we have four adjustable parameters: V, W, R and a. However, there is no *a priori* reason why R and a need be the same for the real and imaginary parts of the potential, so we can have up to six parameters to be determined.

This is the usual form employed in analyses of heavy ion elastic scattering data. However, it is normally found that only three parameters need be varied (for a given system): V, W and a_W (more on this later).

For nucleons and other light ions we usually go a little further:

From theoretical considerations the absorption is expected to be surface peaked, so it has become common to replace $W(r)$ with such a term or to add a surface imaginary term to the volume one.

This may take many forms, e.g. a Gaussian, but the most common is the derivative Woods-Saxon:

$$g(r) = 4W_\mathrm{D}\frac{\mathrm{d}f(r)}{\mathrm{d}r} = -4W_\mathrm{D}\frac{\exp\left(\frac{r-R_\mathrm{D}}{a_\mathrm{D}}\right)}{\left(\exp\left(\frac{r-R_\mathrm{D}}{a_\mathrm{D}}\right)+1\right)^2}$$

This has a peak value of $-W_\mathrm{D}$ at $r = R_\mathrm{D}$ (hence the factor of 4). Note negative real and imaginary potentials are *attractive* and *absorptive* respectively.

Finally, if the projectile has spin there will be (in principle) a spin-orbit potential. While this can affect the elastic scattering cross section angular distribution for nucleons it is “best practice” to include it only when *analysing power* data are available *(polarised beam)*. For heavy ions it is omitted unless this is so.

Usually we take the so-called Thomas form:

$$V_{\mathrm{SO}}(r) = \left(\frac{\hbar}{m_\pi c}\right)^2 (U_{\mathrm{SO}} + iW_{\mathrm{SO}}) \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r}\mathbf{L}\cdot\boldsymbol{\sigma}$$

and $\boldsymbol{\sigma} = 2\,\mathbf{s}$ for nucleons, tritons and ^{3}He,

$\boldsymbol{\sigma} = \mathbf{s}$ for deuterons

In practice, W_{SO} is usually set to zero, since fits to most data sets are not improved by the addition of an imaginary spin-orbit term.

We therefore arrive at the final complete form of the optical model potential for light ions:

$$U(r) = V_{\mathrm{C}} - V f(x_0) + \left(\frac{\hbar}{m_\pi c}\right)^2 V_{\mathrm{SO}} \left(\mathbf{L} \cdot \boldsymbol{\sigma}\right) \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} f(x_{\mathrm{SO}})$$

$$-i \left[W f(x_W) - 4 W_D \frac{\mathrm{d}}{\mathrm{d}x_D} f(x_D)\right]$$

$$f(x_i) = (1 + \exp x_i)^{-1} \quad \text{where} \quad x_i = (r - R_i)/a_i$$

This gives us a total of 12 adjustable parameters (if we fix *R_C*: the results are usually insensitive to the exact value). Rather too many? Sometimes 12 are not enough (we shall come back to this later!)

We can limit the number of variables in practice. We usually parameterise the radius as:

$$R_i = r_i \times \mathrm{A_T}^{1/3}$$

or for heavy ions as:

$$R_i = r_i \times \left(\mathrm{A_I}^{1/3} + \mathrm{A_T}^{1/3}\right)$$

This removes the “trivial” dependence on mass number and we can often fix the r_i for a wide range of nuclei (so although in principle r_0, r_W, r_D and r_{SO} could all have different values they remain fixed). “Typical” values for nucleons are $r_i = 1.15$ fm or 1.25 fm. Often r_0 and r_{SO} will be fixed at the same value, with r_W and r_D fixed at a different value.

It has been found from a large number of analyses that the OMP parameters for a given projectile + target combination vary as a function of bombarding energy. This is true for both nucleons and other light ions and heavy ions, although it can be difficult to prove this for the latter.

The reasons are, however, different. Recall that the formal optical potential is both non-local and L-dependent. While it is always possible to find a local equivalent to such a potential (with some consequences that we will come back to when we look at transfer reactions) this introduces an energy dependence into the local equivalent potential.

For nucleons this energy dependence is dominated by the non-locality introduced by exchange effects.

For heavy ions, exchange effects are essentially negligible, and the energy dependence in the local equivalent potential is due to the non-locality induced by couplings to other reaction channels.

It should be remembered that the empirical OMP that we obtain by fitting data is not the “local equivalent” of the formal optical potential but a purely empirical object, so that some of the energy dependence of the parameters that we find may be intrinsic rather than a consequence of the two sources of non-locality

To recap, we have seen that in its full form the empirical OMP may have up to 12 adjustable parameters. Supposing that we are able to obtain a good description of data by adjusting these parameters (we shall briefly explain how this is done in the next lecture) what have we gained by so doing?

1) By fitting large numbers of data sets for many systems over a wide range of energies we can look for systematic variations with A, Z and incident energy (cf. Kepler and Tycho Brahe's planetary observations).

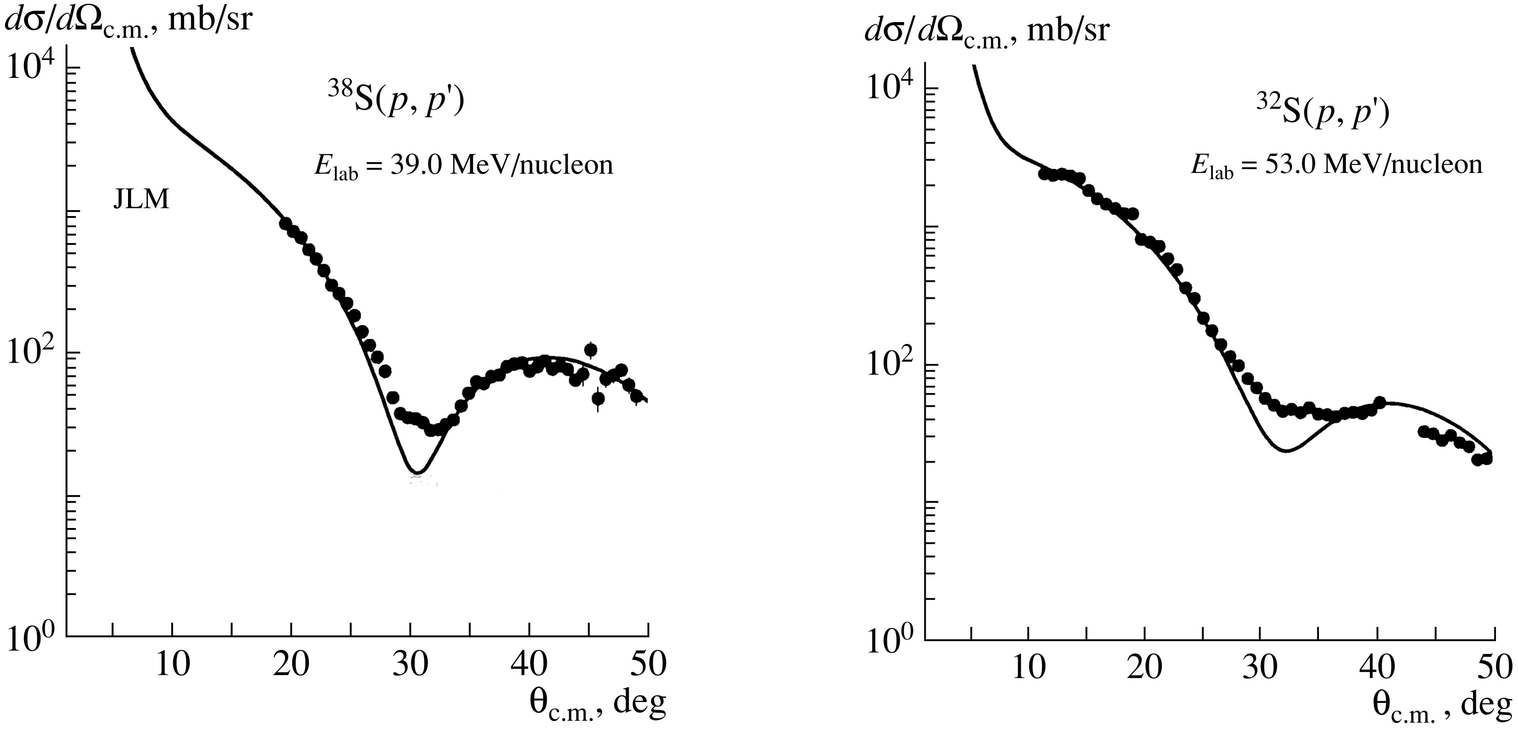
2) For light ions (particularly nucleons) we may obtain “global” parameter sets that fit in an average way large bodies of data for different systems. These are useful in many ways, but can provide greatest insight when they fail badly — this can be an indication of shell effects etc.

3) They are needed as input to other calculations (inelastic scattering and transfer reactions).

It would, of course, be much more satisfying if we could calculate an optical model potential from first principles. We have said that trying to do so using Feshbach theory is virtually impossible. However, it has been done — with considerable success — for nucleons.

While there are more modern theories, the standard to which these are compared is the so-called “JLM” potential: J. P. Jeukenne, A. Lejeune, and C. Mahaux, Phys. Rev. C **16**, 80 (1977). This approach is based on a complex effective nucleon-nucleon interaction and infinite nuclear matter (folded over the target density) so works less well for light targets.

Examples:



Adapted from V. Lapoux, N. Alamanos and E. Khan, Phys. Atomic Nuclei **66**, 1501 (2003)

Description is rather good, but *precise* fits to precise data are not obtainable with JLM or similar potentials based on local density models.

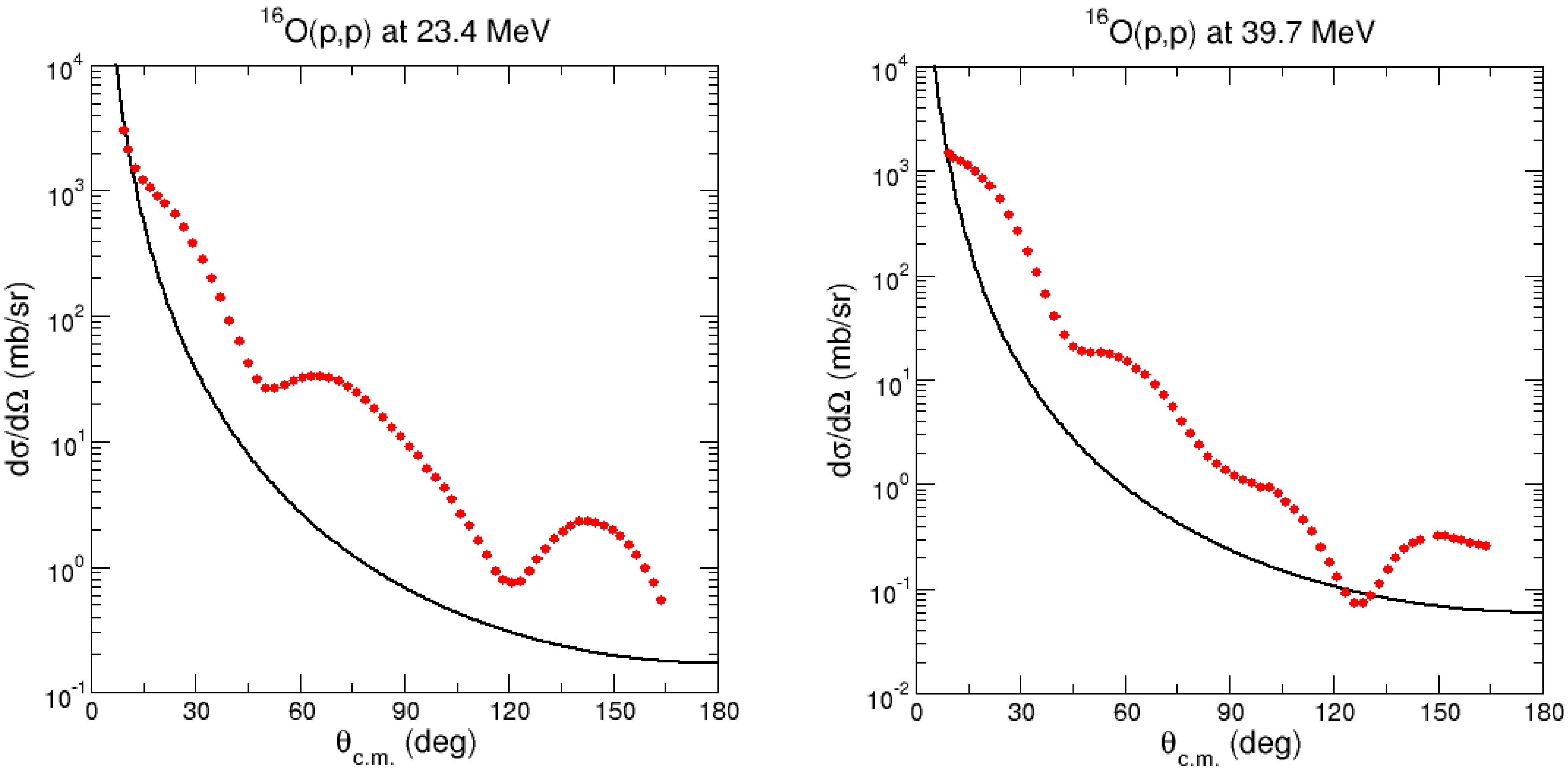
For heavy ions, although various attempts have been made to calculate an OMP the agreement with data is usually worse. The problem is in calculating the imaginary part of the potential. As we shall see later, the real part may be calculated with considerable success for heavy ions, based on clues given by the formal theory.

In the next lecture we shall examine how the optical model is used in practice to analyse light ion elastic scattering data.

Lecture 4: Optical model analysis of light ion elastic scattering

Before proceeding to show how an optical model analysis is performed in practice we will make a few general observations about how light ion elastic scattering data are presented.

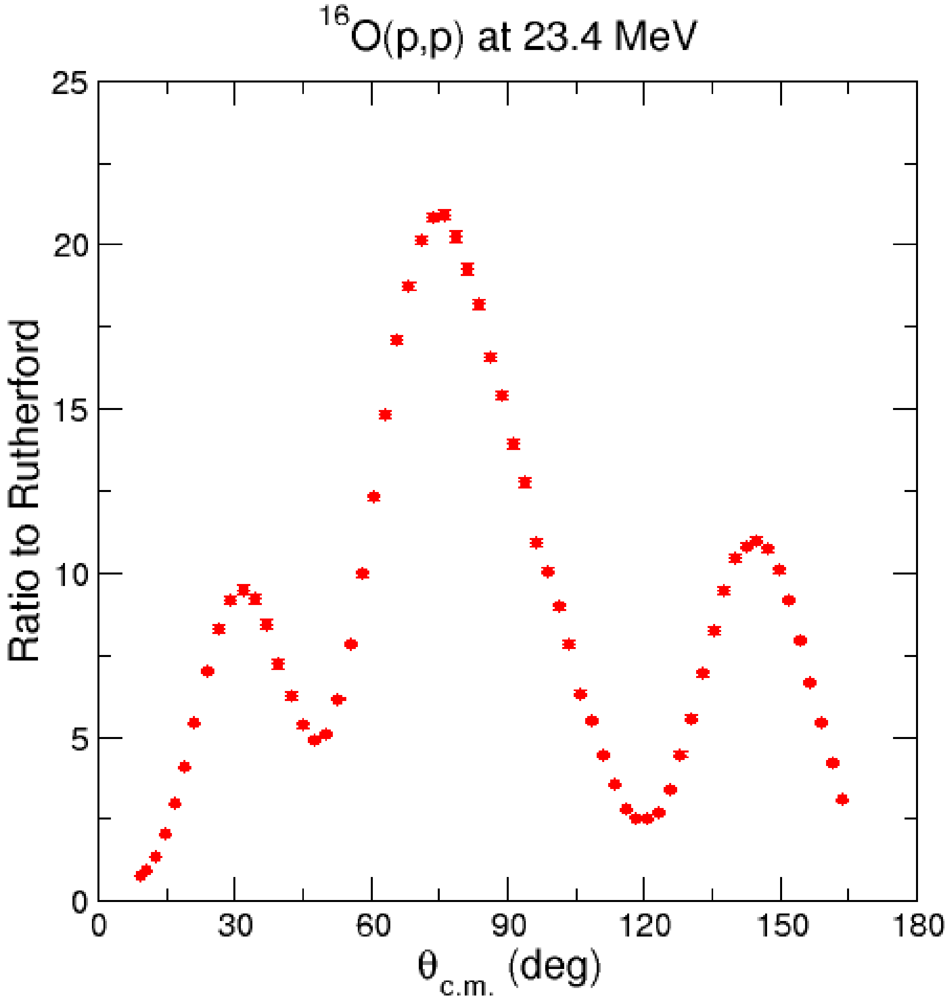
Very often, proton elastic scattering data are plotted on an *absolute* cross section scale, i.e. the angular distribution of the differential cross section is plotted in units of mb/sr:

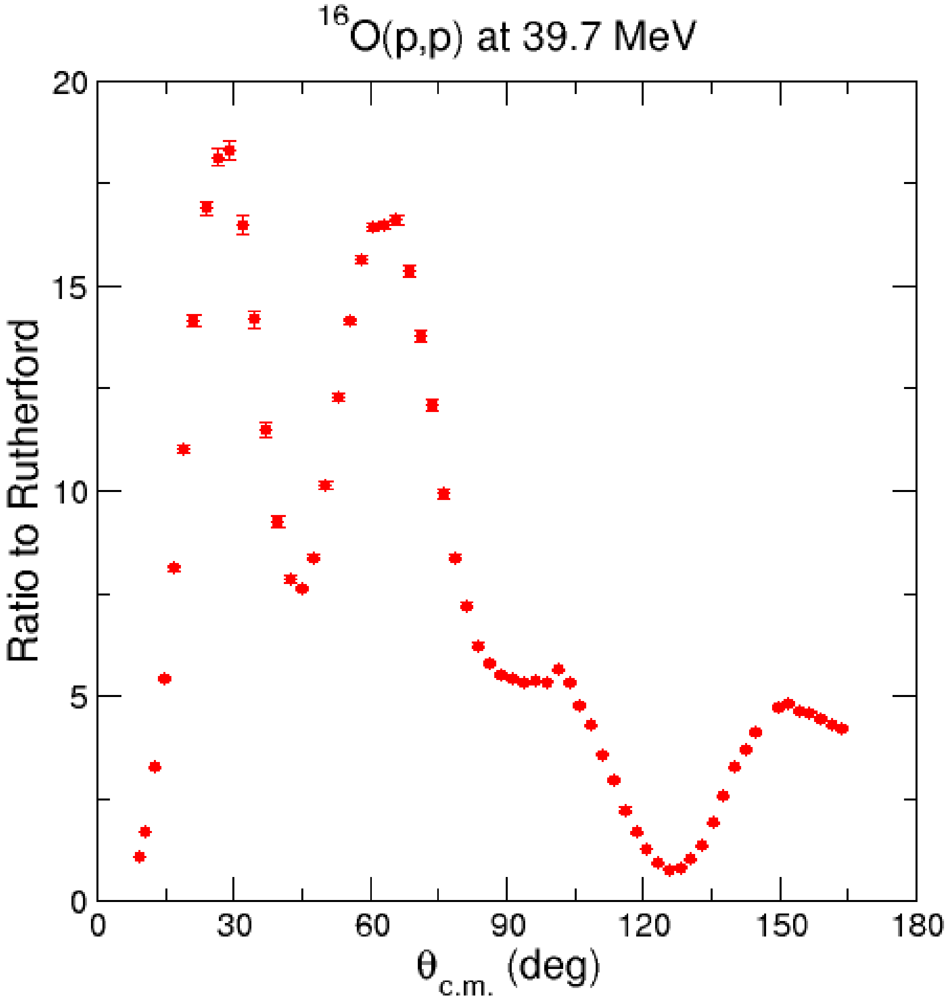


The black curves denote the Rutherford scattering cross section: we are interested in deviations from this …

Data from Phys. Rev. **167**, 908 (1968).

If we divide the measured cross section by the Rutherford cross section we can see the detail we are interested in much more clearly:





This is not usually done for proton scattering, but it should be! It is, however, fairly routine for plotting the elastic scattering of deuterons or heavier ions.

Now (finally) how do we go about performing an optical model analysis?

Even for the relatively simple problem of scattering from a (complex) potential well the equations have to be solved numerically. Our first requirement is thus a code to perform the numerical calculations.

Several are available: ECIS, FRESCO (SFRESCO), HERMES. All allow searching on parameters (we shall return to this shortly).

ECIS and FRESCO are general direct reaction codes which of course includes the ability to perform optical model calculations. HERMES is a specific optical model code. DWBA codes such as DWUCK can also calculate the elastic scattering using the optical model but do not usually allow parameter searches.

Optical model calculations have to be performed numerically (there are no analytic solutions). However, the asymptotic solutions for the wave function *are* known analytically, so we may save some trouble by matching to these at an appropriate radius, the *matching radius*, R_m.

We also need to define a *radial step size* for the numerical integrations that have to be performed, *dr*.

Finally, we truncate the infinite series of partial waves at some finite number, ℓ_{max}.

How do we choose these numbers for a particular case? There is a useful “rule of thumb” which uses the relationship between ℓ and R_ℓ, the classical turning point of the Rutherford orbit:

$$\ell(\ell + 1) = kR_\ell\,(kR_\ell - 2n)$$

$k = 0.2187[m(u)E(MeV)]^{1/2}\ fm^{-1}$
$n = 0.1575\ Z_1Z_2[m(u)/E(MeV)]^{1/2}$

For a given ℓ_{max} we fix R_{m} using this relation, since R_{m} should be at least as large as R_{ℓ} for ℓ_{max}.

To fix ℓ_{max} we use trial and error. For most proton elastic scattering 15 or 20 partial waves are usually sufficient. Larger energies and heavier projectiles and/or targets will need more. It should be large enough that increasing ℓ_{max} makes no difference to our results.

To fix dr, we just choose a value. It should be such that our results do not change if we vary it between reasonable limits. 0.1 fm is a good starting point, although depending on the numerical methods used 0.05 fm or smaller may sometimes be required.

As an example we consider the ^{16}O(p,p) data we saw earlier. There exists in the literature a set of potentials for these data in Phys. Rev. **184**, 1061 (1969), so we begin by repeating these calculations. We use the code FRESCO (www.fresco.org.uk)

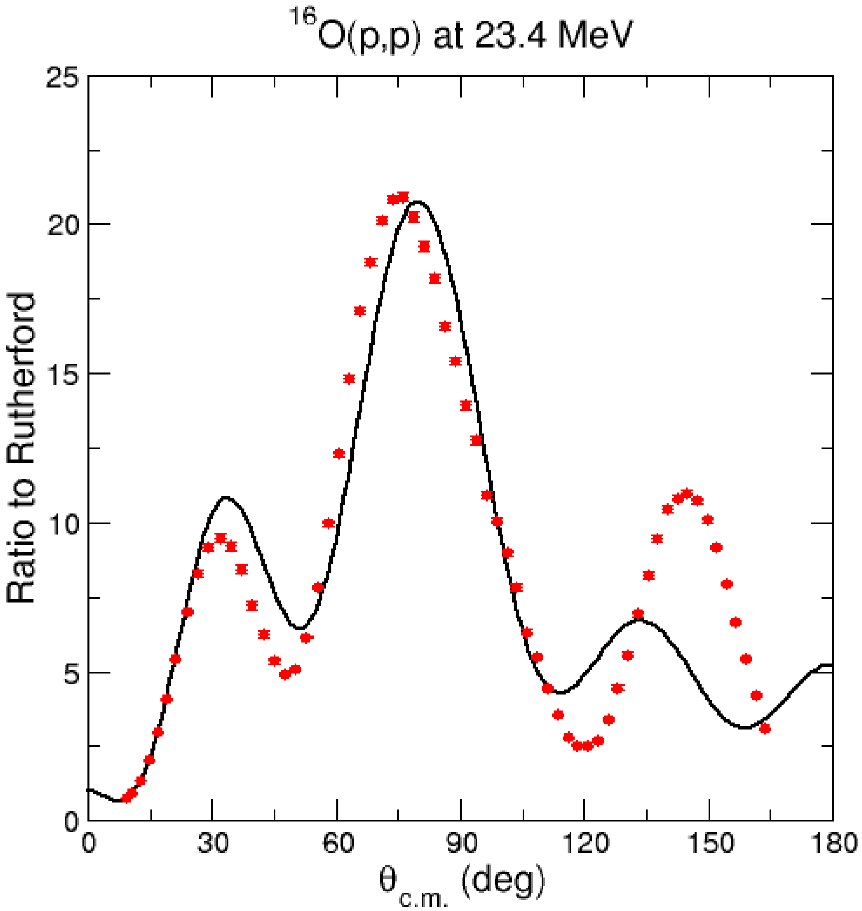
Points to watch in this type of exercise:

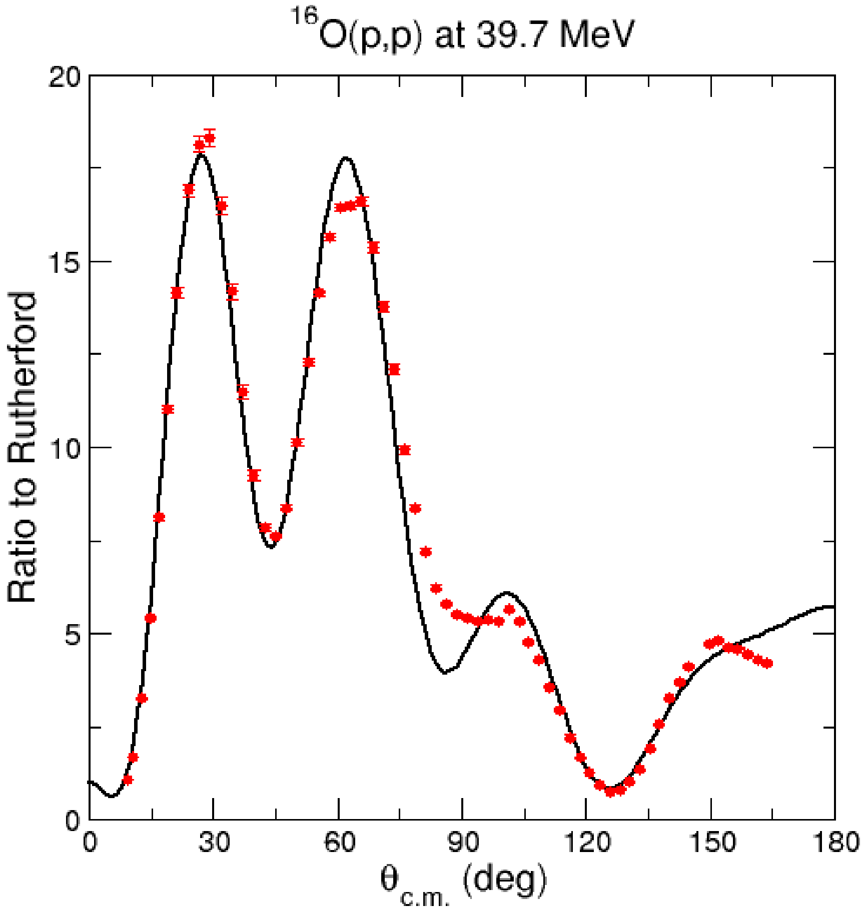
1) Check the definition of any surface terms with that used in the code of your choice. In this case there is a surface imaginary term of Gaussian form. The same thing applies to any spin-orbit terms; sometimes $\mathbf{L} \cdot \mathbf{s}$ is used rather than $\mathbf{L} \cdot \boldsymbol{\sigma}$

2) Do a visual check of the calculation compared to the data; does your calculation match the original? Also check the value of the reaction cross section, if given. Does it match your value? (within small variations due to machine precision).

Now we choose the parameters for the numerical evaluation. Start with *dr* = 0.1 fm. We take ℓ_{max} = 15 as a starting point, which gives values of 15.8 fm and 12.1 fm for the classical turning points at 23.4 and 39.7 MeV respectively. Take R_m = 16 and 15 fm.

How do the results compare to the original calculations and the data?





Visual check is good: compare well to original plots. σ_R also compare well: 499.8 mb and 490.0 mb at 23.4 and 39.7 MeV, compared to 499 mb and 490 mb.

A simple visual check suggests that the description (fit) at 39.7 MeV is much better than that at 23.4 MeV.

This is of course a highly subjective way of assessing the “goodness of fit”. We require an objective way of doing this, ideally one that can be quantified.

Conventionally, “goodness of fit” is assessed in optical model work by the χ^2 value, or most frequently, χ^2 per point, χ^2/N. Strictly speaking, the quantitative values are only meaningful if the errors are purely statistical with a Gaussian distribution (in which case a value of $\chi^2/N = 1.0$ indicates a perfect fit) but they do provide a useful guide and enable comparisons.

The definition of χ^2 is:

$$\chi^2 = \sum_{i=1}^{N} \left[\frac{\sigma_{\mathrm{theor}}(\theta_i) - \sigma_{\mathrm{expt}}(\theta_i)}{\Delta\sigma_{\mathrm{expt}}(\theta_i)} \right]^2$$

In our cases, χ^2/N is 916.7 and 154.2 for E_p = 23.4 and 39.7 MeV respectively; not close to 1.0!

While this is not good (in principle) there are three reasons for this:

1) The data are precise (error bars of less than ± 2%)

2) The original fits were constrained to have the same “geometry” (i.e. radii and diffuseness) at all energies

3) Polarisation data were also included in the fit

However, *precise* data deserve *precise* fits; if not there is information contained in them that we are missing. Can we improve on these fits, and how do we go about it?

We must search on the parameters and try to minimise χ^2. However, the OMP used here has 9 parameters (12 at 39.7 MeV); searching on all of them at once is not a good idea!

Two reasons:

1) Purely practical; can take a long time to minimise χ^2 when varying so many parameters

2) The parameters are not, in fact, independent, there are correlations between some of them.

It is also not usually a good idea to adjust the spin-orbit potential parameters without polarisation data.

We will see if we can improve significantly the fit at 23.4 MeV. Since the “geometry” was fixed for a wide range of energies, try searching on that.

Initial values are:

V = 47.25 MeV, r_V = 1.142 fm, a_V = 0.726 fm

W_d = 7.06 MeV, r_d = 1.268 fm, a_d = 0.980 fm

V_{SO} = 4.09 MeV, r_{SO} = 1.114 fm, a_{SO} = 0.585 fm

We start with a_V. Search on this parameter alone:

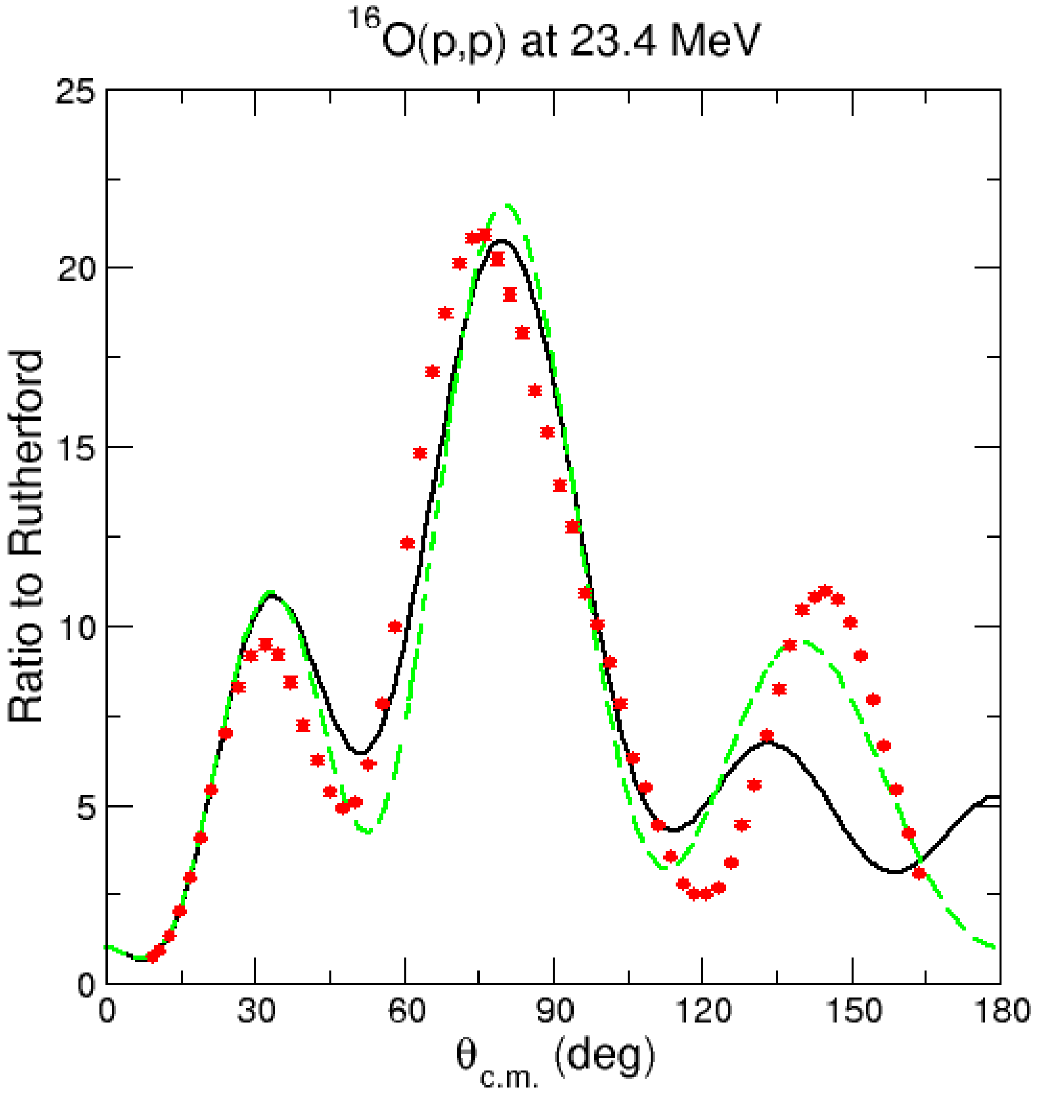
SFRESCO (searching version of FRESCO) minimises χ^2 and yields a minimum value of χ^2/N = 717 for a_V = 0.660 fm.

Now search on r_V. χ^2/N = 708, r_V = 1.136 fm.

Search on r_d. χ^2/N = 594, r_d = 1.365 fm.

Search on a_d. χ^2/N = 583, a_d = 0.955 fm.

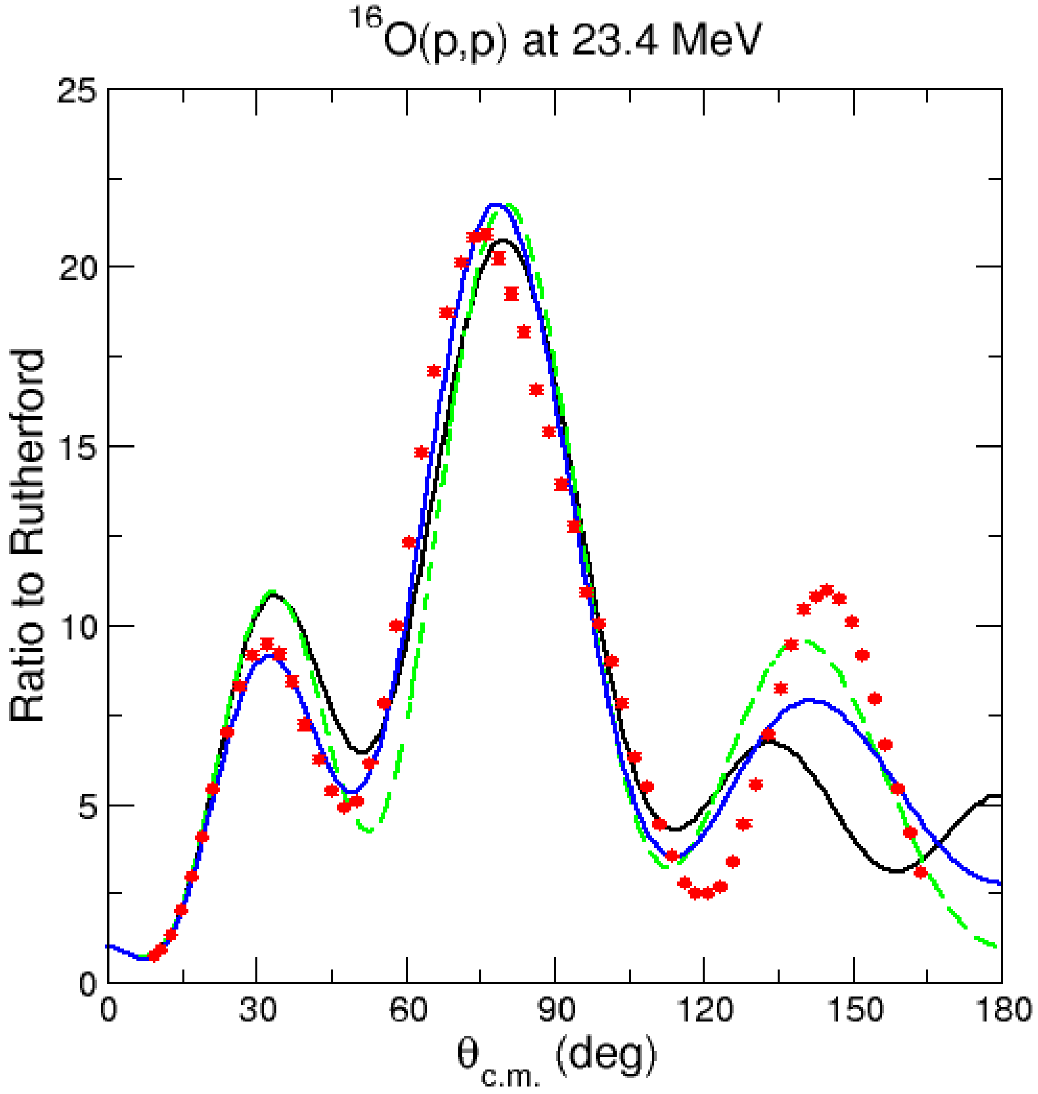
We could continue the search, but we have already improved the χ^2/N value considerably. How does the angular distribution compare now?



Improved, but we are clearly still missing something in our fit. More parameter searching required or a deficiency in our model?

σ_R has not changed significantly though: 514 mb cf. 500 mb.

If we search on all 6 parameters of the central part of the potential what do we find? χ^2/N is now reduced to 343, σ_R = 554 mb.



We are still some way from a “good” description of the data, but at least the first peak is now well described …

Agreement is better, but parameters are now a little strained:

V = 64.65 MeV, r_V = 0.9095 fm, a_V = 0.841 fm
W_d = 6.55 MeV, r_d = 1.363 fm, a_d = 1.095 fm

(recall that we have not searched on spin-orbit potential).

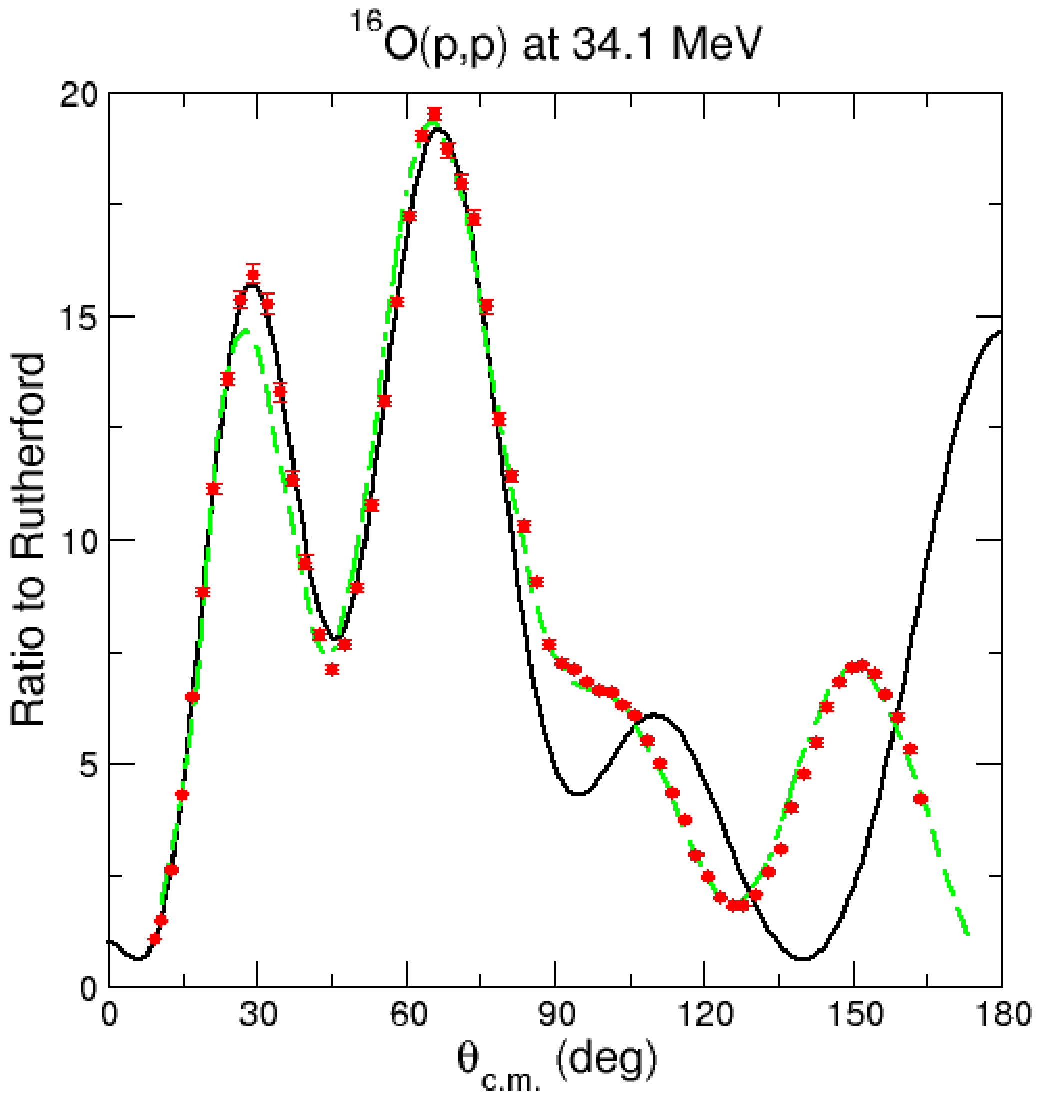
This is about as far as we can go without introducing yet more parameters (we could add a volume absorption term, as at 39.7 MeV, but recall the well known quote of von Neumann!). What does this comparative failure to fit the data well tell us?

1) Fitting precise data precisely can be difficult!

2) As we saw previously, the OMP is intrinsically non-local (and L-dependent). While it is always possible to find a local equivalent to this N-L potential it may not be (and in general will not be) parameterisable with “standard” potential forms.

3) Linked to 2), strong couplings may induce effective potentials that cannot be satisfactorily modelled with standard potential forms.

However, if we allow an L-dependent potential we can fit these data well:



Slightly different energy from same data set. OMP of usual form but with additional (complex) L-dependent terms. χ^2/N for L-dependent OMP is 8.79, cf. 1160 for original L-independent OMP!

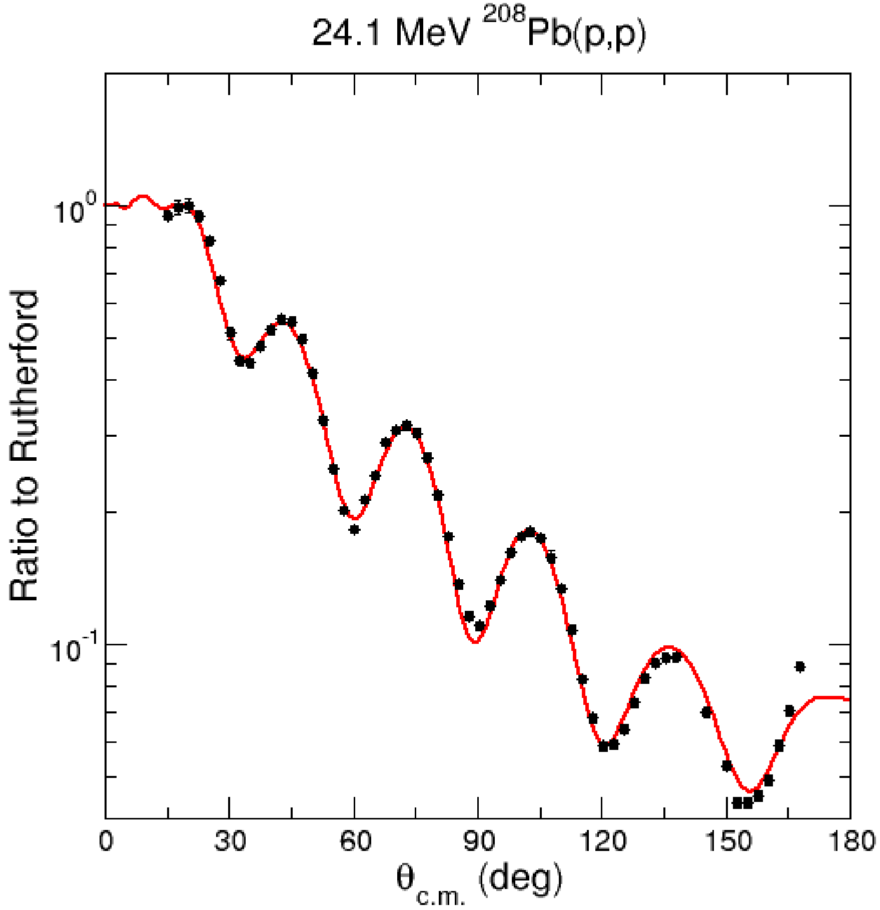
L-dependent fit from Kobos and Mackintosh, J. Phys. G **5**, 97 (1979).

Evidence for L-dependence seems compelling. It does introduce extra parameters, but improvement in quality of fit is dramatic.

The good fit does not tell us what causes the L-dependence, but at least we now know that the OMP for this system must have this characteristic.

Also suggests that standard OMP forms can be inadequate for proton scattering from light targets. Could also be a signature of strong coupling effects on elastic scattering from other reaction channels (giant resonances, (p,d) pickup etc.)

However, just to show that all is not doom and gloom, the optical model in its standard form with standard Woods-Saxon potentials can describe data for elastic scattering of protons from heavier targets very well:


24.1 MeV ^{208}Pb(p,p)

Standard 9-parameter Woods-Saxon potential can fit these data well: $\chi^2/N = 5.57$! (only 6 parameters searched on; spin-orbit potential was held fixed).

Similar fits obtained over a wide incident energy range.

Data from: W. T. H. van Oers *et al*., Phys. Rev. C **10**, 307 (1974)

A word about potential ambiguities …

The OMP is not normally uniquely determined by the data. The multi-dimensional χ^2 space will in general have local minima as well as the overall or "global" minimum. These minima are often quite broad in certain directions, so some parameters are not well determined; there are *ambiguities*.

OMP ambiguities are of two basic types: continuous and discrete. Continuous ambiguities arise due to the broadness of the "valleys" or minima in the χ^2 space. The best known continuous ambiguity is:

$$V r_V^n = \text{constant}$$

with $n \approx 2$

Any change in V can be compensated for by a corresponding change in r_V (within certain limits, up to about 10 %).

Similar ambiguities exist for W and a_W and there are undoubtedly more involving three parameters.

Discrete ambiguities are somewhat different. It is found that if V is steadily increased and the other parameters readjusted to optimise the fit χ^2 passes through a whole series of minima. Basically, this is due to an additional half-wave fitting inside the well than for the next shallowest potential. This leads to “families” of deep and shallow potentials.

Lecture 5: Optical model analysis of heavy ion elastic scattering

While the theory we use is still the same – the optical model does not differ for light and heavy ions – there are some important differences in the analysis techniques for heavy ion elastic scattering.

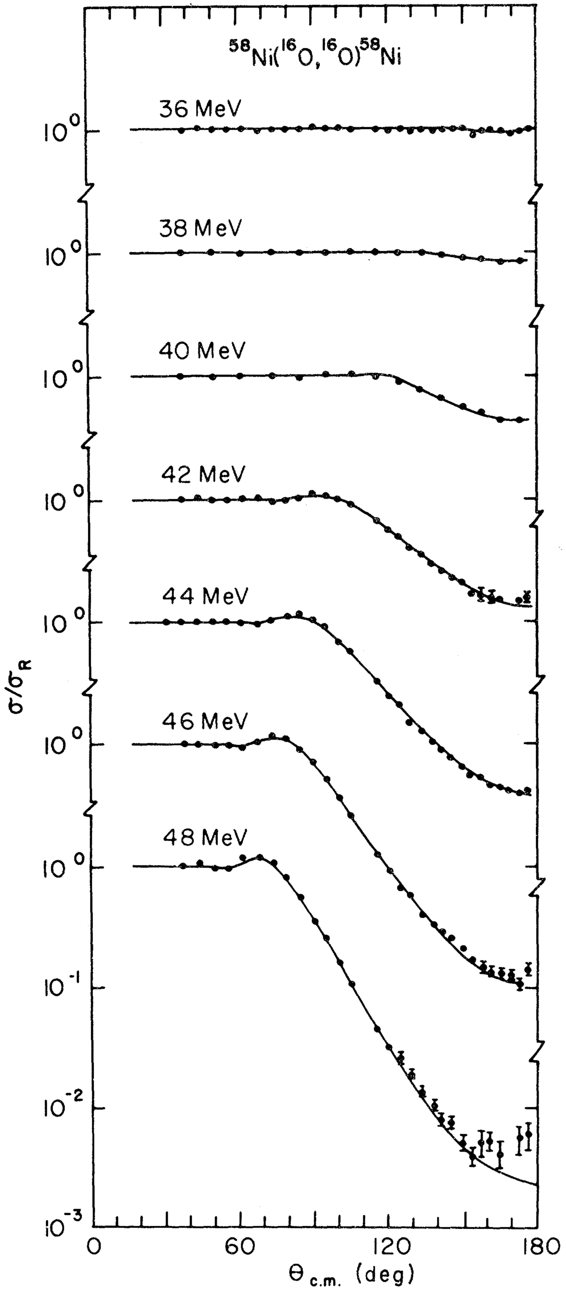
The target nucleus is more or less transparent to nucleons, deuterons and to a lesser extent the other light ions (t, ^{3}He, α).

In contrast, heavy ions are strongly absorbed and the target is more or less opaque or even “black”.

This has important implications for optical model analyses: we are no longer sensitive at all to the potential in the nuclear interior. Heavy ion elastic scattering data only determine the optical potential in a relatively narrow region at the nuclear surface so that the ambiguities in the OMP extracted from data are even more pronounced than for light ions.

There will thus usually be many “families” of OMPs that fit a given data set, each with its own continuous ambiguities. This should always be borne in mind when speaking of “the OMP” for heavy ion elastic scattering.

What do “typical” heavy ion elastic scattering angular distributions look like? Example: ^{16}O + ^{58}Ni



Two things to note:

1) The evolution of the shape of the AD as the incident energy is reduced.

2) The cross section scale is logarithmic

Taken from West, Kemper and Fletcher, Phys. Rev. C **11**, 859 (1975)

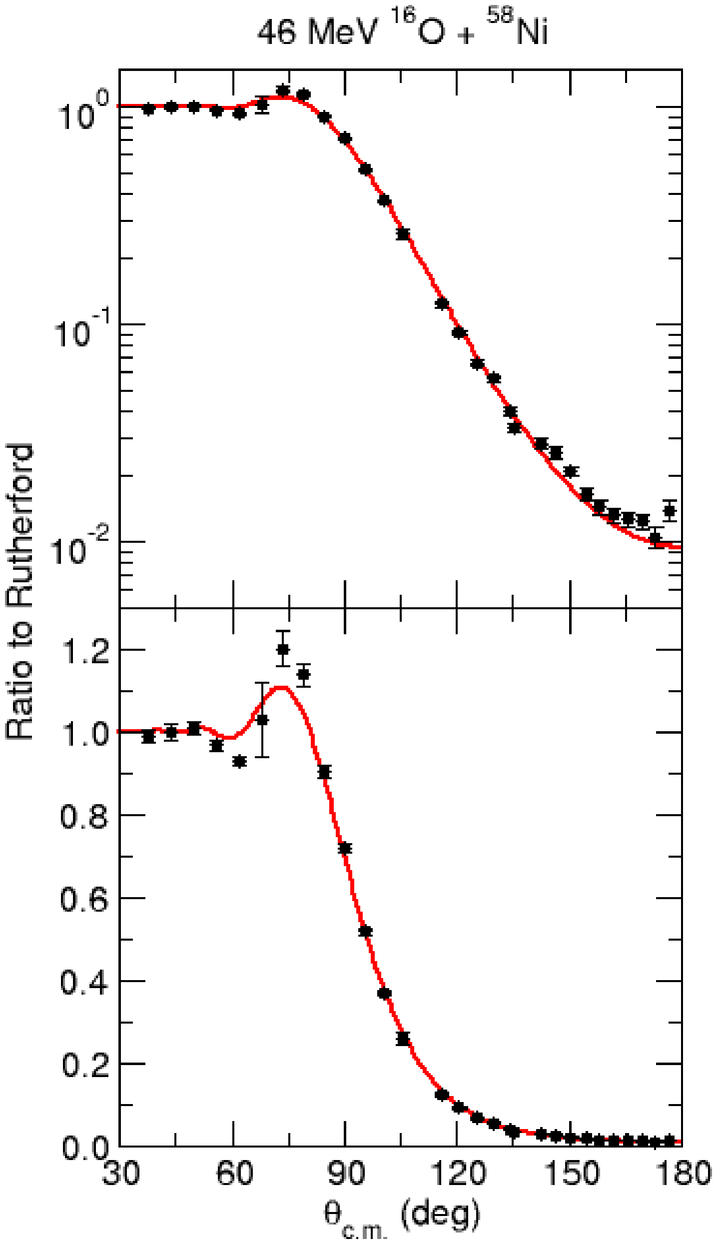
Why is the logarithmic cross section scale significant?

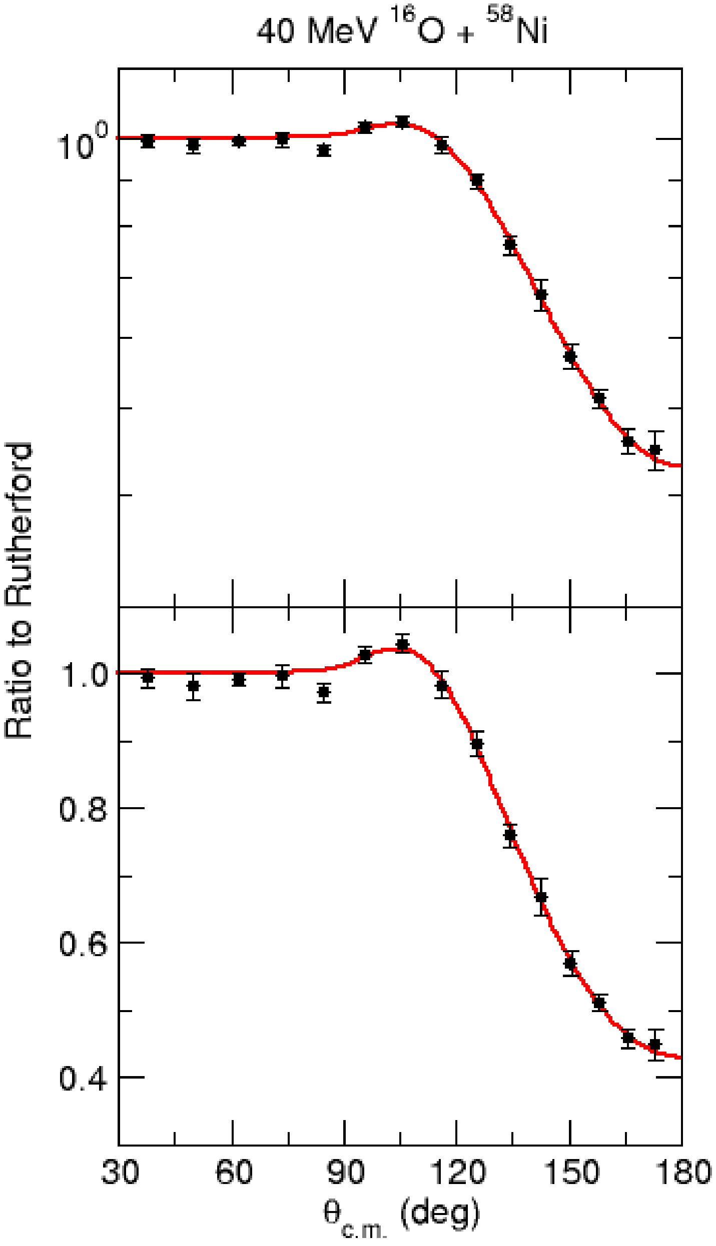
If we only plot our data in this fashion we may miss some significant information:

"*A particularly sensitive part of the angular distribution is the oscillatory region just before the exponential fall below the Rutherford cross section. (The details of this region may be overlooked if, as is often done, the ratio-to-Rutherford cross section is shown on a semi-logarithmic plot. It is more revealing to use a linear plot.)*" J. B. Ball *et al.*, Nucl. Phys. A **252**, 208 (1975).

See also: G. R. Satchler, Phys. Lett. B **55**, 167 (1975).

We demonstrate what this means using the ^{16}O + ^{58}Ni data:





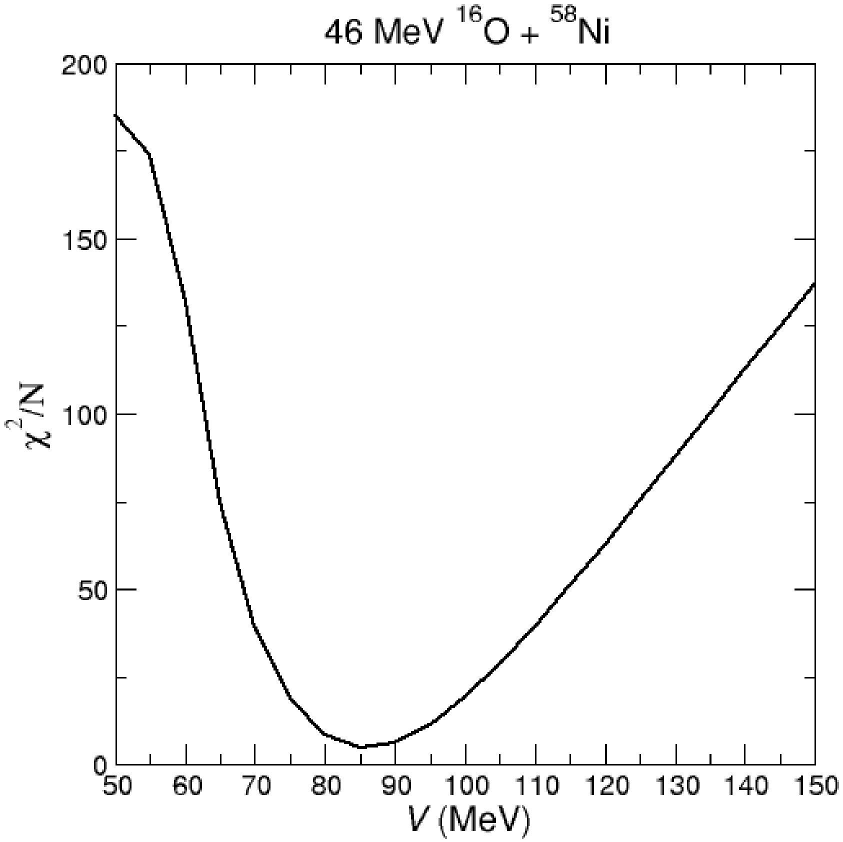
Fits are from the original publication: “geometry” is fixed at $r_0 = 1.22 \times (16^{1/3} + 58^{1/3})$ fm, $a_0 = 0.50$ fm for both real and imaginary parts (so vary *V* and *W* to obtain best fit).

While the fits are rather good – $\chi^2/N = 0.71$ and 4.77 at 40 and 46 MeV respectively – the linear plot shows room for improvement at 46 MeV.

V and *W* are: 90.5 MeV and 7.26 MeV at 40 MeV
85.7 MeV and 33.3 MeV at 46 MeV

How well defined are these values, and how can we find out?

Since the geometry is fixed here, we have an ideal case for a *grid search* on *V* and/or *W*: we fix *V* at some value and then optimise the fit (in this case, we search on *W* to minimise χ^2) and note the value of χ^2. We then change *V* by some incremental value and repeat the exercise. We then end up with a plot like this:



This is a somewhat artificial case but the general principle holds. If the data are not very precise or if there is little structure to fit (i.e. close to the Coulomb barrier) then the minimum will not be so well defined.

The constraint that the real and imaginary geometries are the same is rather artificial for heavy ion scattering. In general it will not be possible to obtain good fits to data under this assumption (there is no real physical reason for doing so anyway).

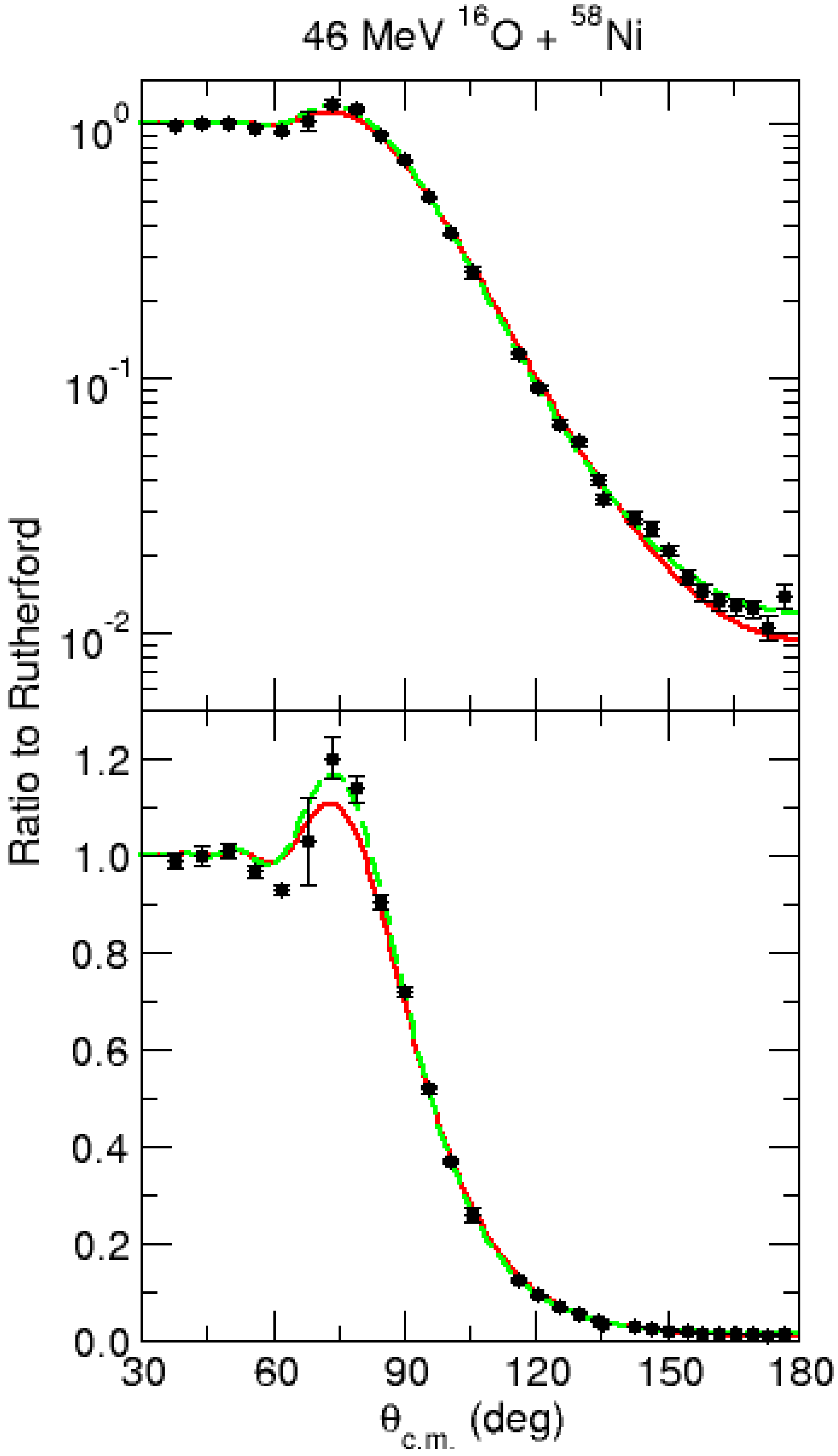
If we remove this constraint can we significantly improve the fit at 46 MeV ? Yes, but only at the cost of unphysical OMP parameters:

Fix $r_{\mathrm{C}} = r_V = r_W = 1.3 \times (16^{1/3} + 58^{1/3})$ fm

Search on the other parameters: $V = 29.8$ MeV, $a_V = 0.55$ fm

$W = 44.4$ MeV, $a_W = 0.324$ fm

$\chi^2/\mathrm{N} = 2.51$ (c.f. previous value of 4.77)



The fit is very good but the real depth is unphysically low, especially compared to the imaginary depth. The imaginary diffuseness is also a little low (normally expect values around 0.5 to 0.6 fm).

We seem to have got stuck in a “shallow” real potential family. Is there any way of avoiding this?

Yes, and at the same time we can at least place the real part of the OMP on a sounder theoretical basis.

Feshbach theory gives us a clue. This suggests that we may write the real part of the OMP in the following schematic fashion:

$$V(r) = V_f(r) + \Delta V(r)$$

where $V_f(r)$ is the so-called *double-folding potential* and $\Delta V(r)$ is the real part of the *dynamic polarisation potential (DPP)*. The double-folding potential is obtained by integrating — “folding” — an effective nucleon-nucleon force over the matter densities of the projectile and target.

The DPP arises due to the effects of coupling to the non-elastic channels (inelastic excitations, transfers).

In fact, the DPP also has an imaginary term and we could also write $W(r)$ in a similar schematic fashion. The imaginary part of the DPP accounts for the absorption induced by the couplings to other direct channels, with the equivalent of $V_f(r)$ being an imaginary potential “inside” the Coulomb barrier that accounts for absorption due to fusion.

However, the DPP is essentially impossible to calculate *ab initio*, so this formal exercise does not seem to have helped us much so far.

Nevertheless, it can be turned to practical use. We make no attempt to calculate the imaginary part of the OMP from first principles and instead keep the empirical (usually Woods-Saxon) form with its three adjustable parameters.

For the real part, we assume that $\Delta V(r)$ is small enough compared to $V_\mathrm{f}(r)$ that we may safely neglect it. We therefore arrive at what we might call a “semi-microscopic” OMP of the following form:

$$U(r) = N_\mathrm{R}\, V_\mathrm{f}(r) + \mathrm{i}\, W(r)$$

where N_R is a normalisation factor.

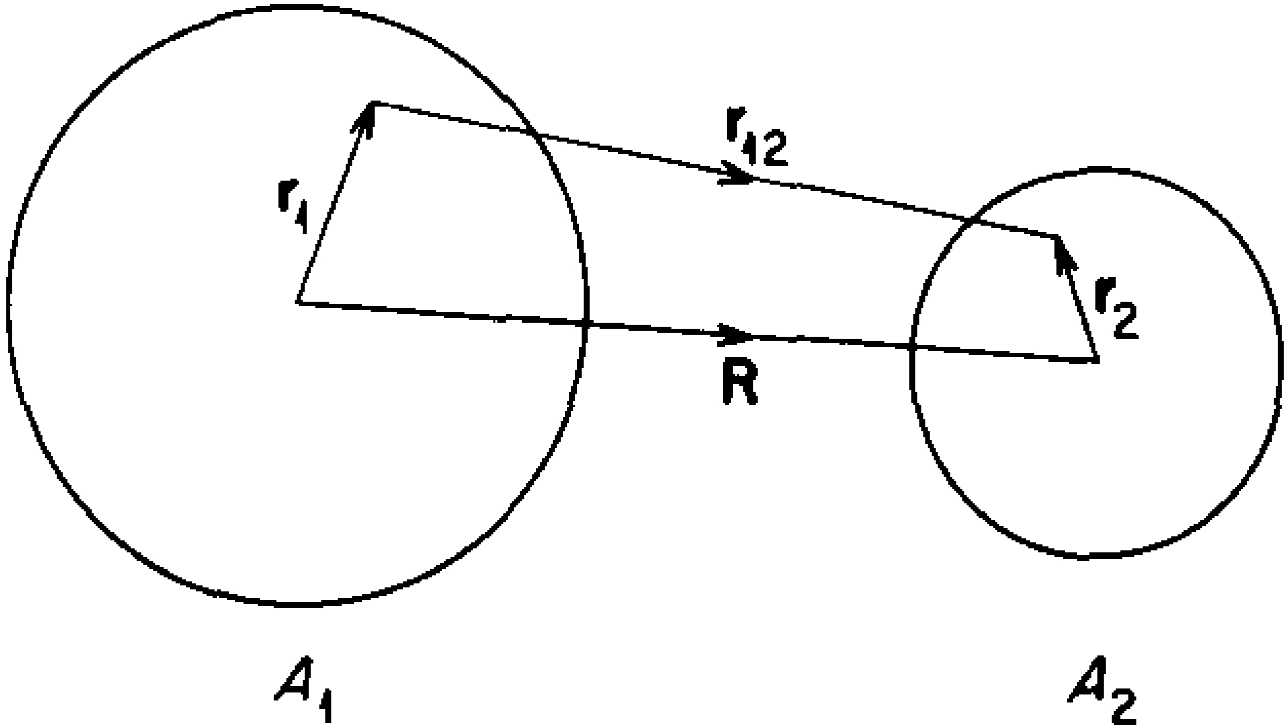
If our assumption is good then N_R should be close to 1.0. A large body of data was analysed using this model by Satchler and Love, Phys. Rep. **55**, 183 (1979) and they found a mean value of:

$$N_R \approx 1.11 \pm 0.13$$

so the assumption seems to be valid for a wide range of systems (we shall discuss the exceptions later).

The use of the double-folding model to calculate the real part of the OMP at least fixes the shape and, to some extent, the depth of the real potential, leaving us with a maximum of 4 adjustable parameters.

How, then, do we calculate $V_{\mathrm{f}}(r)$? For interacting nuclei with mass numbers A_1 and A_2 we define the following coordinate system:



$V_{\mathrm{f}}(r)$ is then calculated as:

$$V_f(\mathbf{R}) = \int \mathrm{d}\mathbf{r}_1 \int \mathrm{d}\mathbf{r}_2 \rho_1(\mathbf{r}_1)\rho_2(\mathbf{r}_2)v(\mathbf{r}_{12} = \mathbf{R} + \mathbf{r}_2 - \mathbf{r}_1)$$

We need the densities, $\rho_1(\mathbf{r}_1)$ and $\rho_2(\mathbf{r}_2)$ and the effective nucleon-nucleon interaction. The densities are the nuclear *matter* densities. These can be obtained by calculation, e.g. the shell model, or can be derived from empirical *charge* densities (from e.g. electron scattering).

The charge density is first converted to the proton density by unfolding the proton charge distribution. The neutron matter density is then obtained by assuming that:

$$\rho_n = (N/Z)\,\rho_p$$

this is only a reasonable approximation if $N \approx Z$

The most popular choice for the effective interaction is the so-called M3Y. This was used by Satchler and Love in their extensive study. There is a more modern form known as the BDM3Y1 which also depends on the nuclear density but which gives similar results.

Finally, we need a code to calculate the potential for us. Several exist, with DFPOT being the most readily available (Comput. Phys. Comm. **25**, 125 (1982)).

Armed with our double-folding real potential, how well can we describe the 46 MeV ^{16}O + ^{58}Ni data? Can we improve on the original fit without ending up with unphysical parameters? Yes, if we proceed with care …

As a first guess, we start with the same imaginary potential parameters as in the original Woods-Saxon fit and set $N_\mathrm{R} = 1.0$. We keep r_W fixed at 1.22 fm and search on the remaining three parameters: N_R, W and a_W. We arrive at the following "best fit" solution:

$N_\mathrm{R} = 1.41$, $W = 92.8$ MeV, $r_W = 1.22$ fm, $a_W = 0.368$ fm

with $\chi^2/\mathrm{N} = 2.79$. The imaginary well depth still seems rather large (although the double-folding real potential is very deep, 907 MeV at $r = 0$ fm) but a_W is rather more reasonable than our previous attempt. The AD is visually indistinguishable from the Woods-Saxon "best fit" (the dashed green curve on the plot).

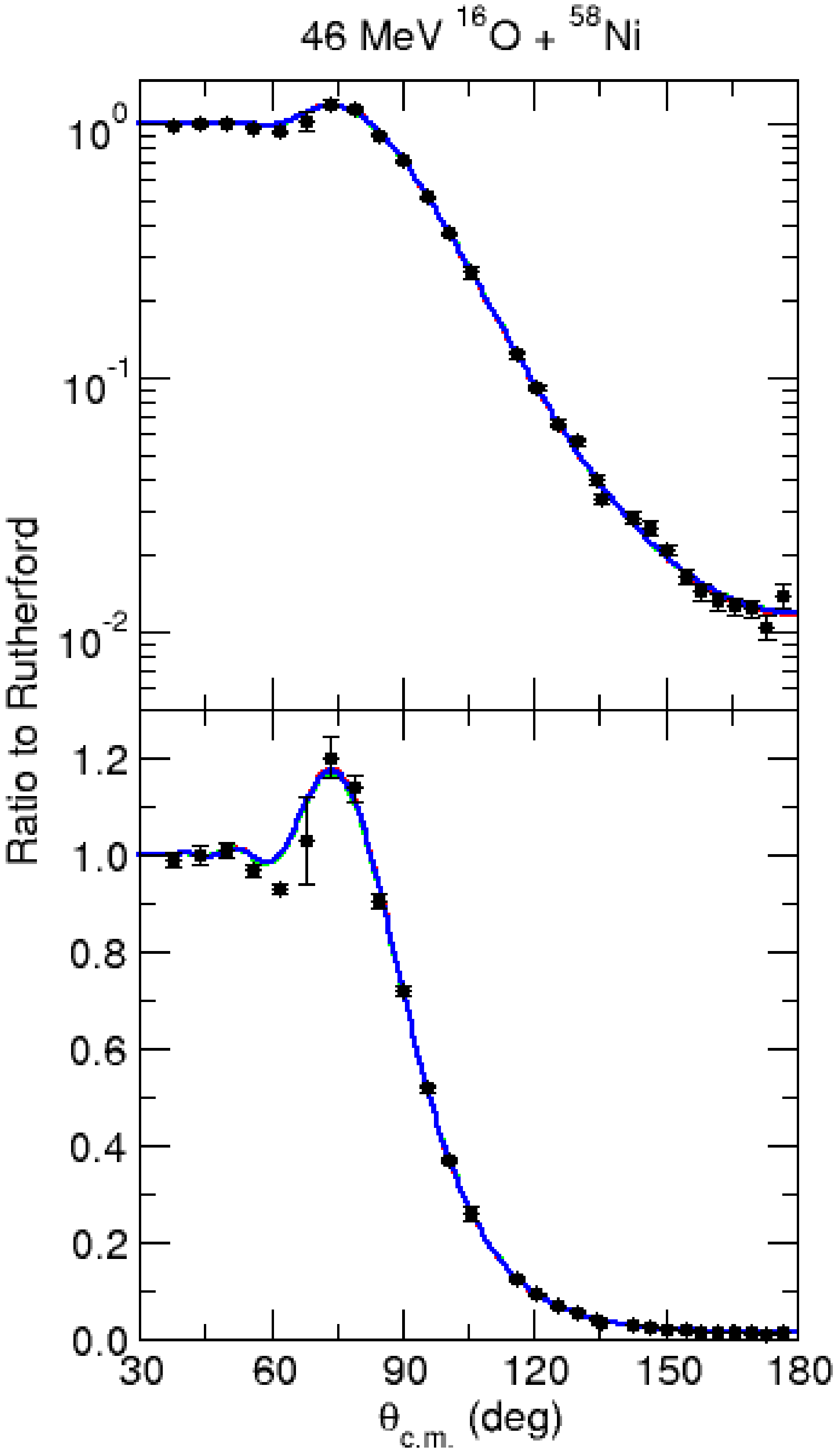
If we fix r_W = 1.3 fm we arrive at the following best fit OMP parameters:

N_R = 1.40, W = 24.0 MeV, r_W = 1.3 fm, a_W = 0.362 fm

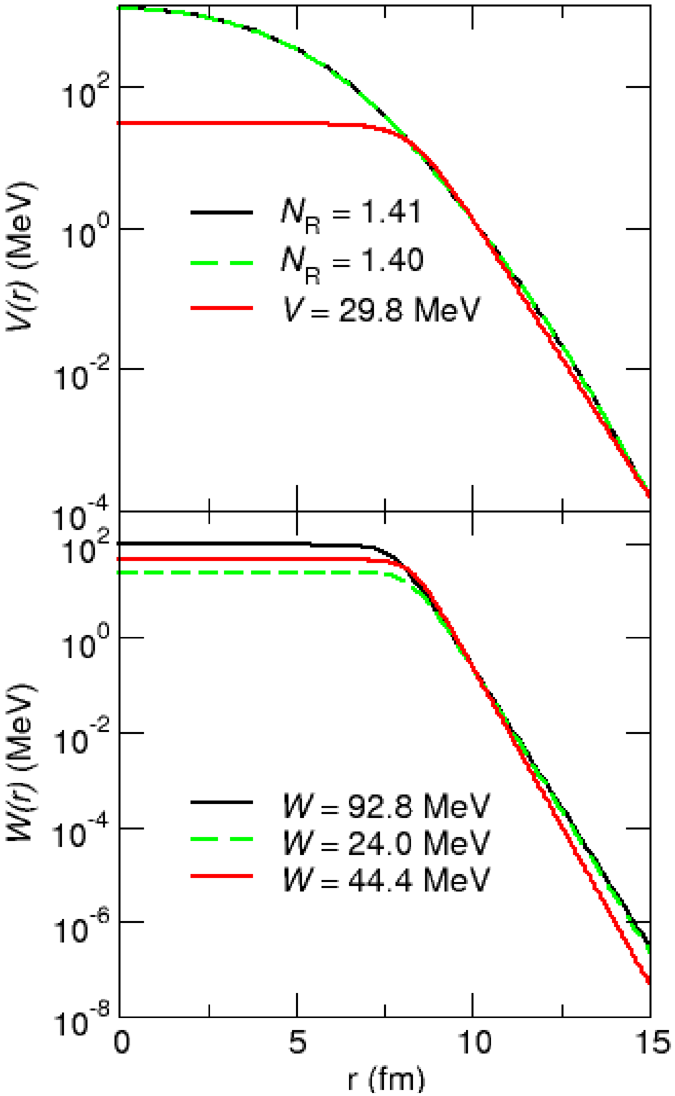
W is now much more reasonable and a_W has hardly changed. χ^2/N = 2.71 and the AD is again visually indistinguishable from the previous best fits.

Everything now seems to be under control, but …
Didn't we say that we ought to find that $N_\mathrm{R} \approx 1.0$?
1.4 is definitely not ≈ 1.0! What's going on?

Before we answer that question, let us compare the various “best fit” results.



There are in fact three curves in the figures, the “best fit” Woods-Saxon and the two DF fits (with $r_W = 1.22$ and 1.3 fm). How do the OMPs compare?



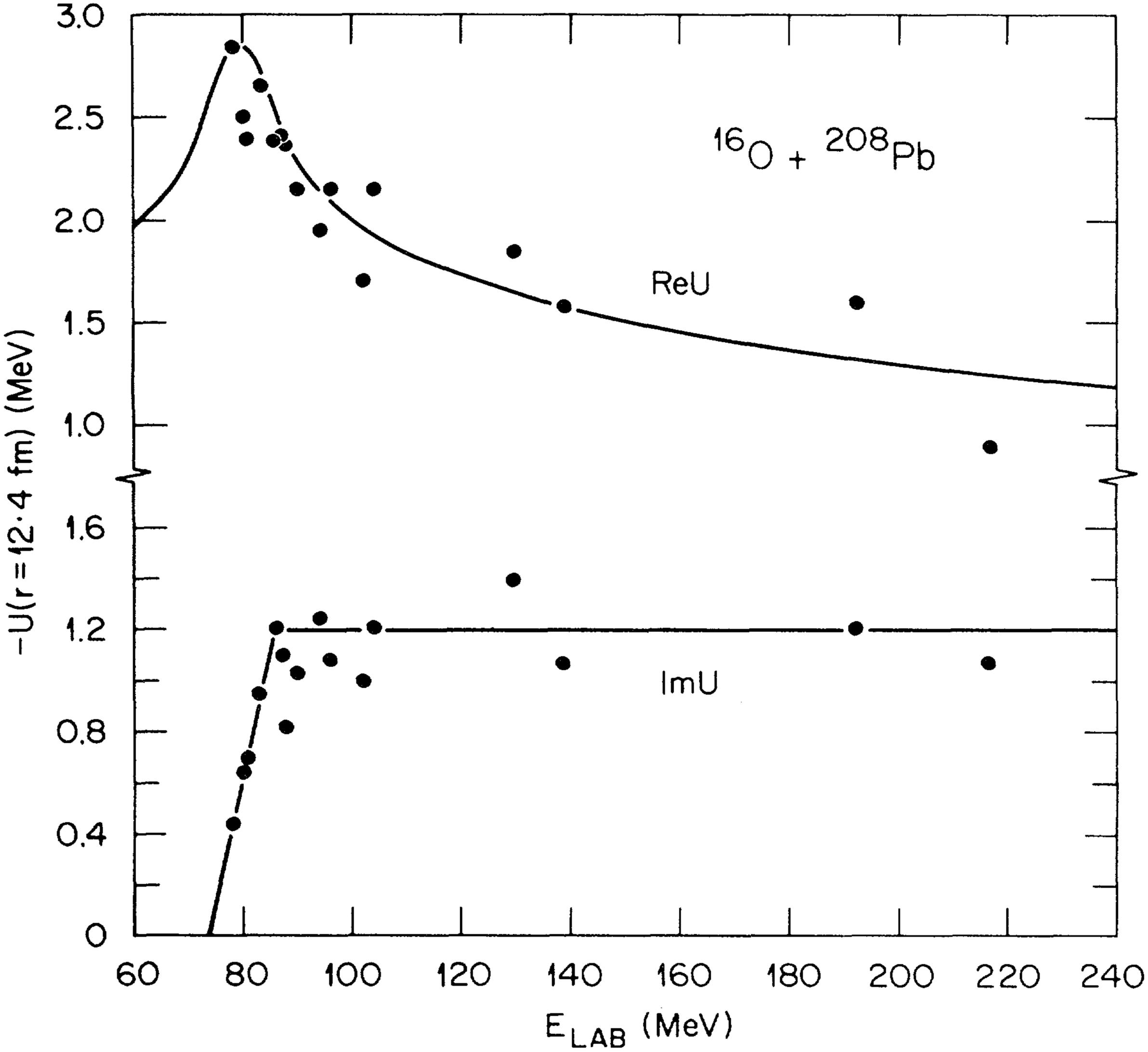
Our assumption that Δ*V*(*r*) is small enough to be neglected has broken down. It is found that for incident energies close to the Coulomb barrier the real DPP can become substantial compared to the double-folding potential, at least in the surface.

The real DPP is linked to the imaginary one via a dispersion relation:

$$\begin{aligned} V(\mathbf{r}, \mathbf{r}'; E) &= V_0(\mathbf{r}, \mathbf{r}') + \frac{\mathrm{P}}{\pi} \int_0^\infty \frac{W(\mathbf{r}, \mathbf{r}'; E')}{E' - E} \, dE' \\ &= V_0 + \Delta V(E), \end{aligned}$$

so as *W*(*r*) varies rapidly near the barrier, so does Δ*V*(*r*).

First demonstrated for the ^{16}O + ^{208}Pb system:

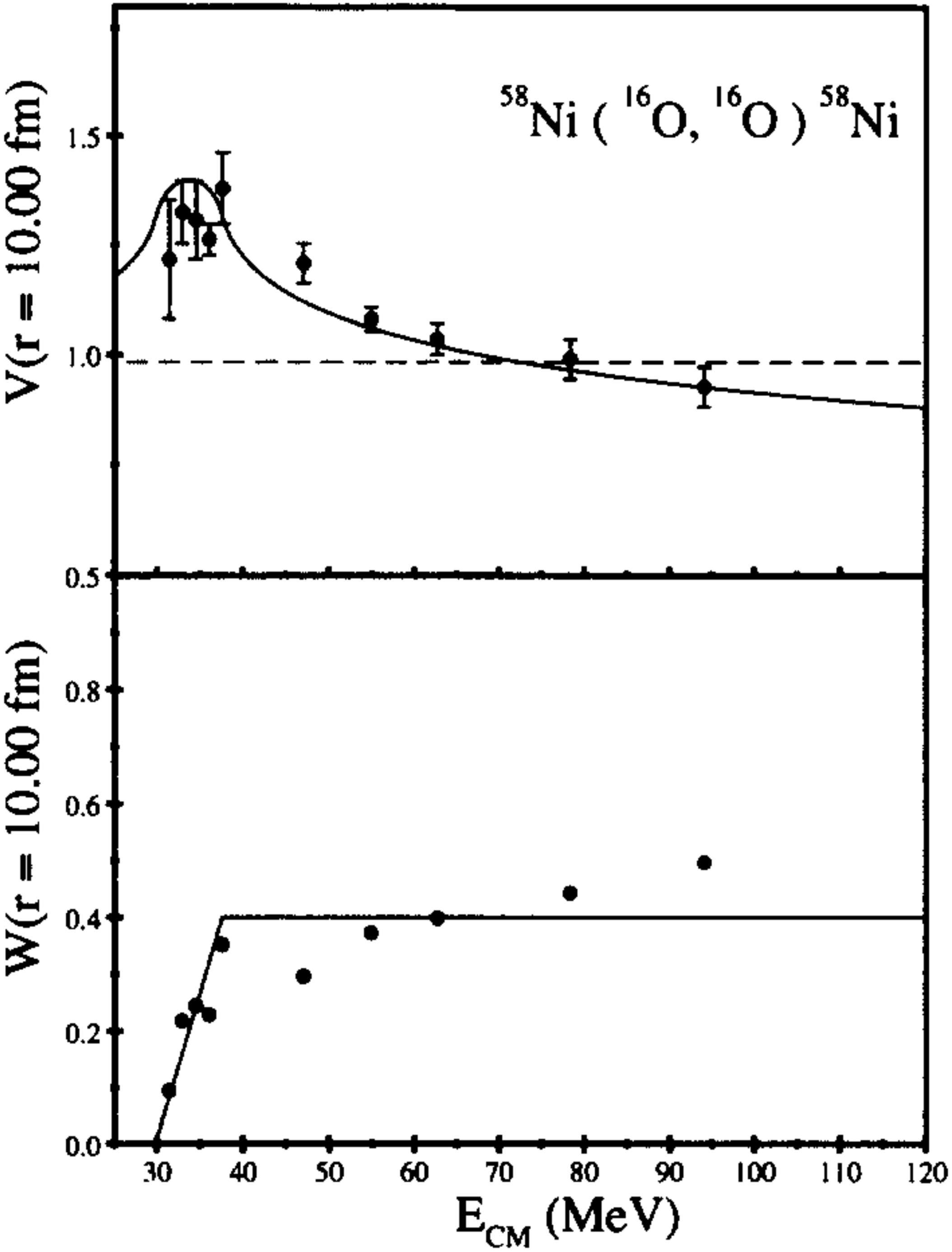


Nagarajan, Mahaux and Satchler, Phys. Rev. Lett. **54**, 1146 (1985)

The effect should, in principle, be universal, but it can be difficult to demonstrate unambiguously.

We can simulate the effect of $\Delta V(r)$ by the factor N_R because for heavy ions the data are only sensitive to the potential around the surface; the effective real potential induced by couplings is peaked at the surface.

The ^{16}O + ^{58}Ni system is one case where the effect is large enough to be clearly defined as an energy variation in the surface strength of the potentials. They also follow a dispersion relation quite well …



Keeley *et al.*, Nucl. Phys. A **582**, 314 (1995).

The double folding model also breaks down under a different set of circumstances. It was found to fail, in that values of $N_R \approx 0.5 – 0.6$ were required, for systems involving ^{6}Li, ^{7}Li and ^{9}Be as projectiles.

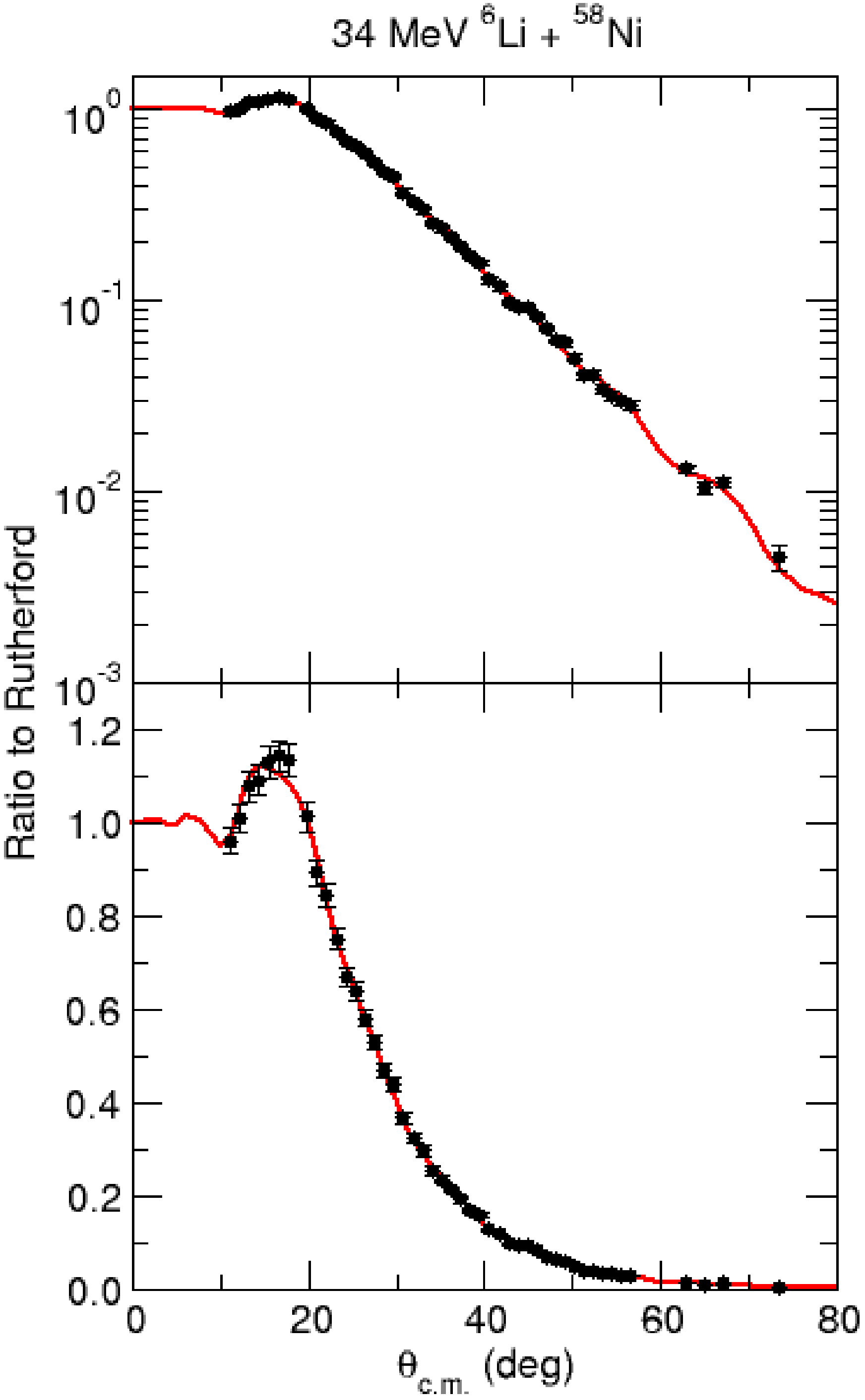
These nuclei have one important property in common, viz. they are all weakly bound, i.e. they have low (~ 1 – 2 MeV) thresholds against breakup into 2 or more fragments.

As we shall see in the next lecture, this was the key to explaining the apparent failure of the double folding model.

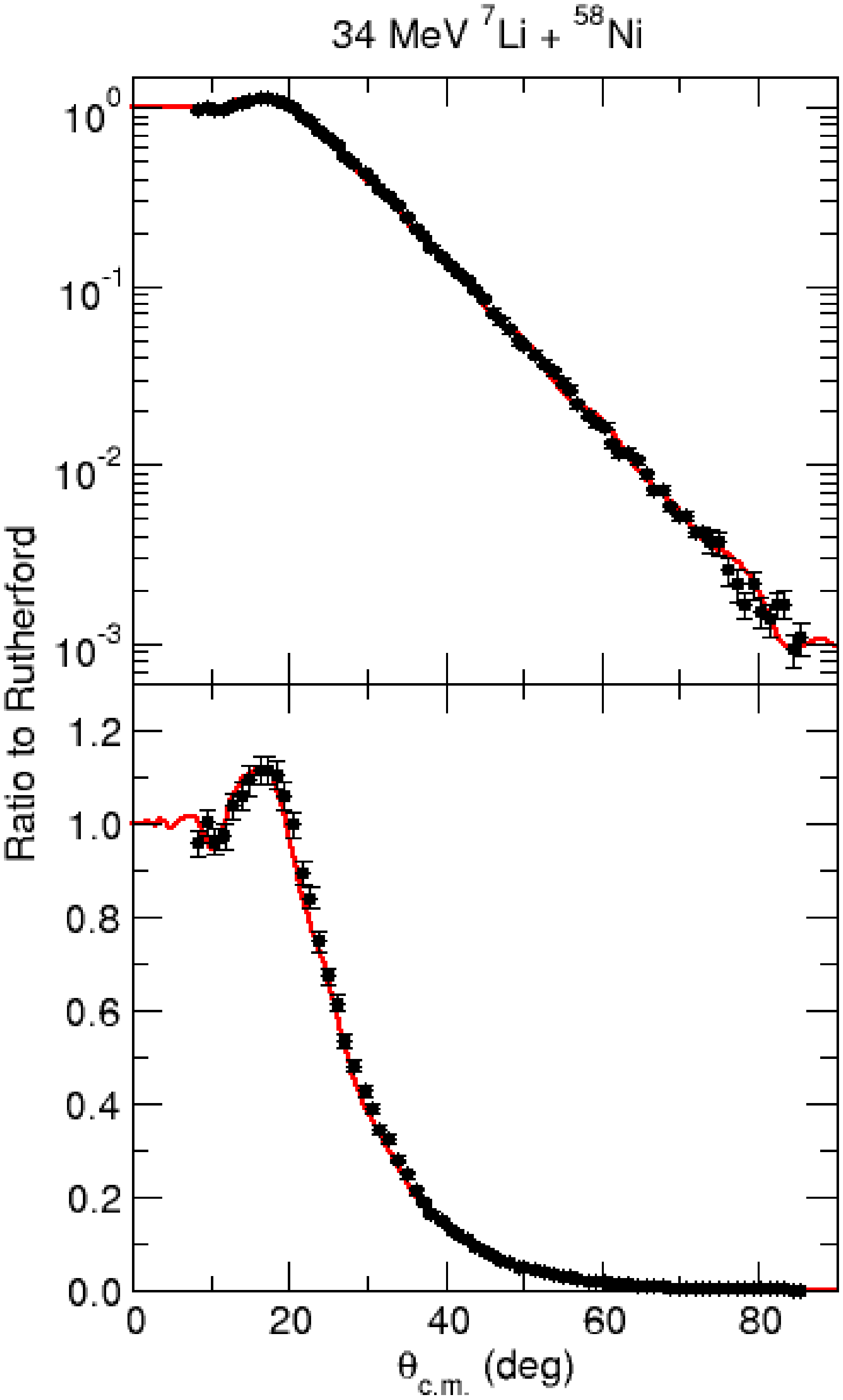
Lecture 6: Optical model analysis of heavy ion elastic scattering continued

We saw last time that the “semi microscopic” OMP consisting of a renormalised double folding real part and a Woods-Saxon imaginary part was able to describe well a wide range of data with $N_R \approx 1.0$, with the exception of energies close to the Coulomb barrier where $\Delta V(r)$ can be large compared to $V_f(r)$.

Other exceptions are data for ^{6,7}Li and ^{9}Be projectiles:



Data: Veal *et al.*, Phys. Rev. C **60**, 064003 (1999)



Data: Glover *et al*., Nucl. Phys. A **341**, 137 (1980)

The best fit OMP parameters are:

^{6}Li: $N_{\mathrm{R}} = 0.60$,
$W = 10.96$ MeV, $r_W = 1.3$ fm, $a_W = 0.693$ fm

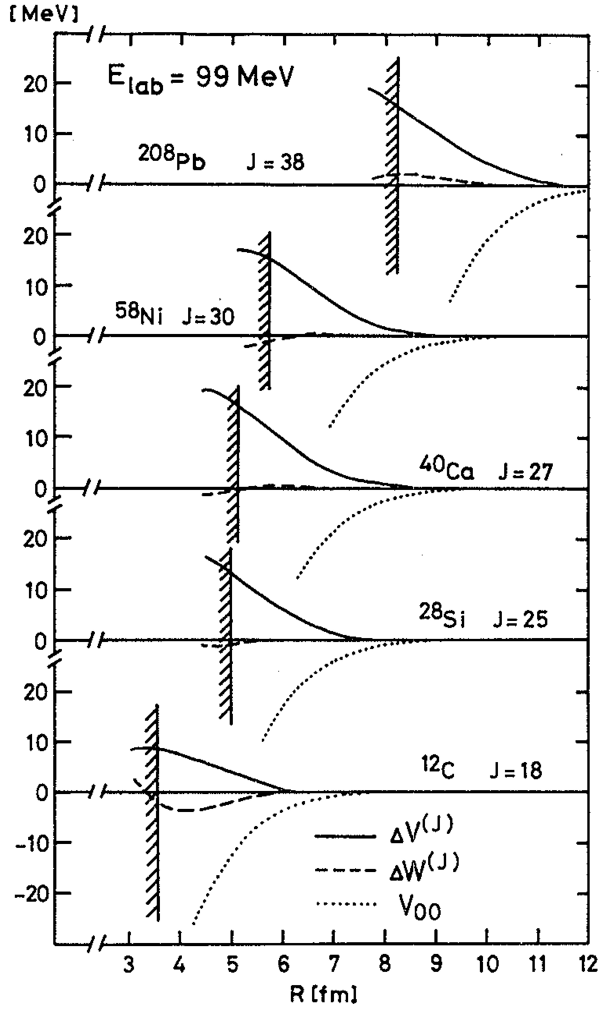
^{7}Li: $N_{\mathrm{R}} = 0.76$,
$W = 12.78$ MeV, $r_W = 1.3$ fm, $a_W = 0.692$ fm

We find that N_{R} is significantly smaller than 1.0 for these systems. This is an almost general phenomenon for ^{6,7}Li projectiles (and ^{9}Be too).

What causes it?

Again, our assumption that $\Delta V(r)$ is small enough compared to $V_f(r)$ to be neglected breaks down.

This time, it is due to a *repulsive* real DPP induced by the breakup of the projectile.

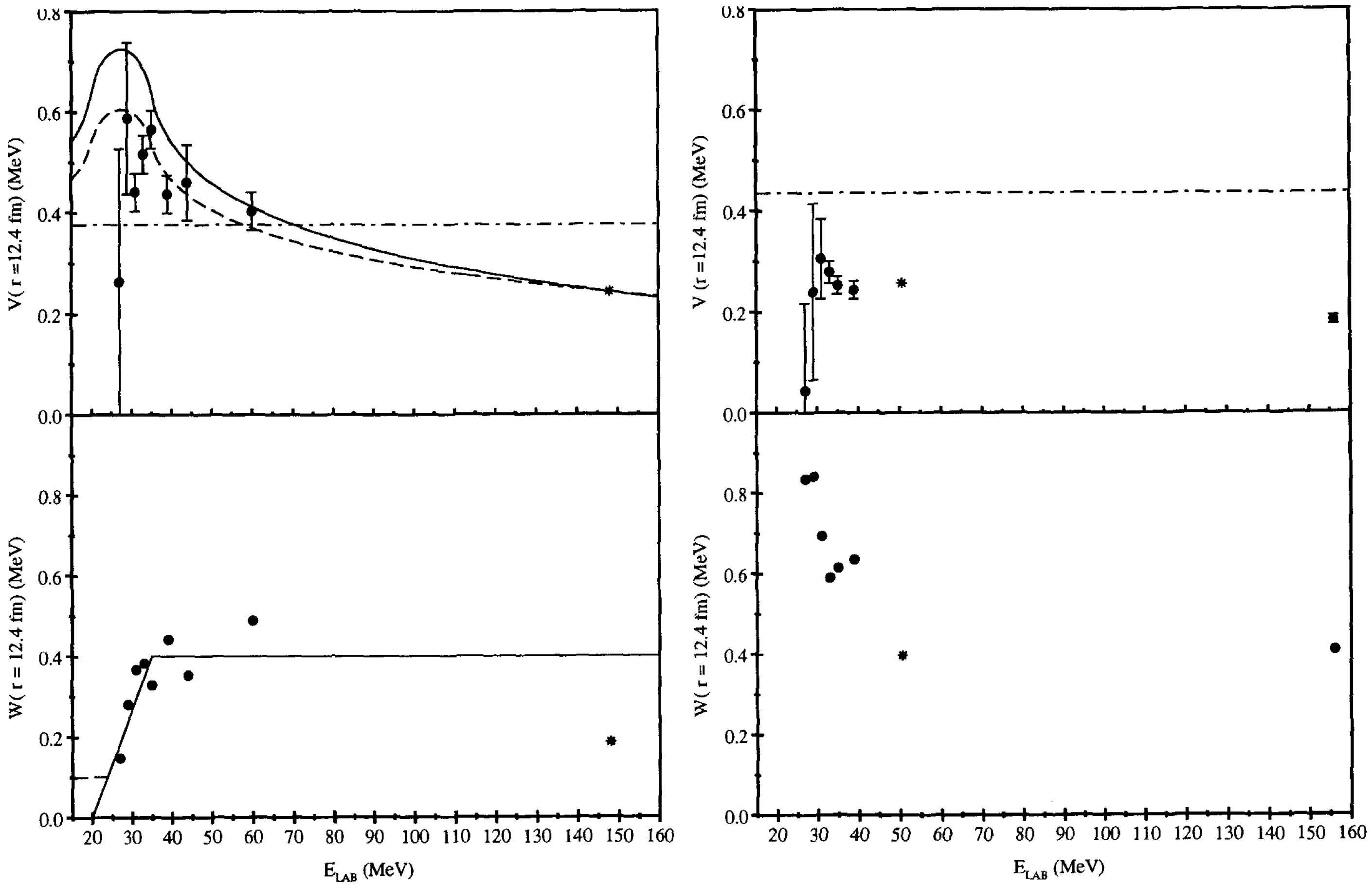


This has been shown explicitly: e.g. Sakuragi, Yahiro and Kamimura, Prog. Theor. Phys. Suppl. **89**, 136 (1986)

Note that $\Delta W(r)$ is very small in the surface, although this does depend on the details of the calculation. $\Delta V(r)$ is always positive in this region.

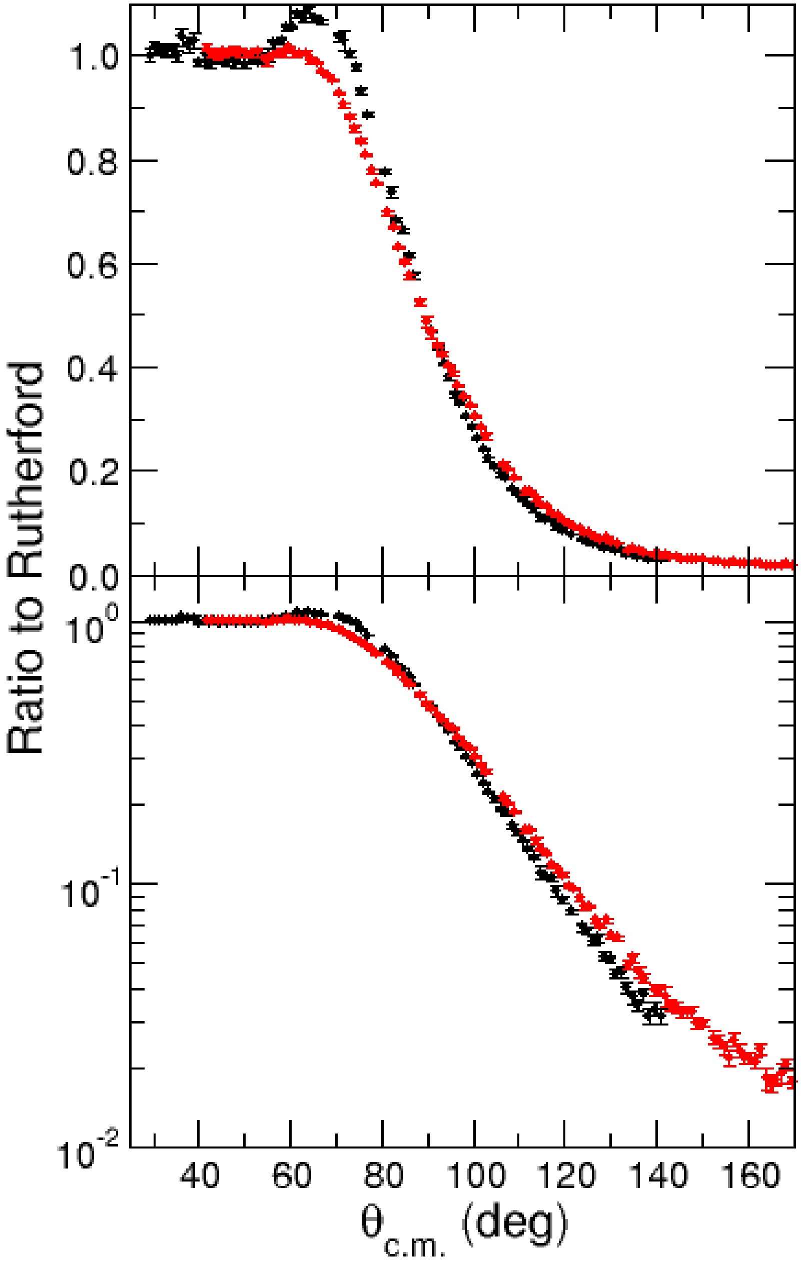
Thus, to simulate $\Delta V(r)$ for breakup we need $N_R < 1.0$

However, it does not always hold. For ^{6,7}Li elastic scattering from a ^{208}Pb target at energies close to the Coulomb barrier we find that while N_R is still around 0.6 for ^{6}Li it becomes larger than 1.0 for ^{7}Li (like a "normal" heavy ion …)



Taken from Keeley *et al.*, Nucl. Phys. A **571**, 326 (1994)

We can even detect the difference in the data:



The plot shows data for ^{7}Li + ^{208}Pb (black) and ^{6}Li + ^{208}Pb (red) at incident energies of 35 MeV. The Coulomb rainbow peak is clearly much more pronounced for ^{7}Li than it is for ^{6}Li. This is reflected in the greater surface strength of the ^{7}Li real potential.

This behaviour was traced to the 1 MeV difference in breakup threshold between ^{6}Li and ^{7}Li but the effect becomes less prominent for lighter targets and larger incident energies.

This brings us quite nicely to the final part of this lecture: why study elastic scattering?

There are two answers to this question:

1) Since elastic scattering is always present, it is a vital ingredient in the analysis of other direct reactions. We need optical potentials that describe the appropriate elastic scattering as input to any analysis of inelastic scattering and transfer reactions.

2) For its own sake. Elastic scattering can provide information on the structure of the colliding nuclei itself. Coupling effects are also worthy of study.

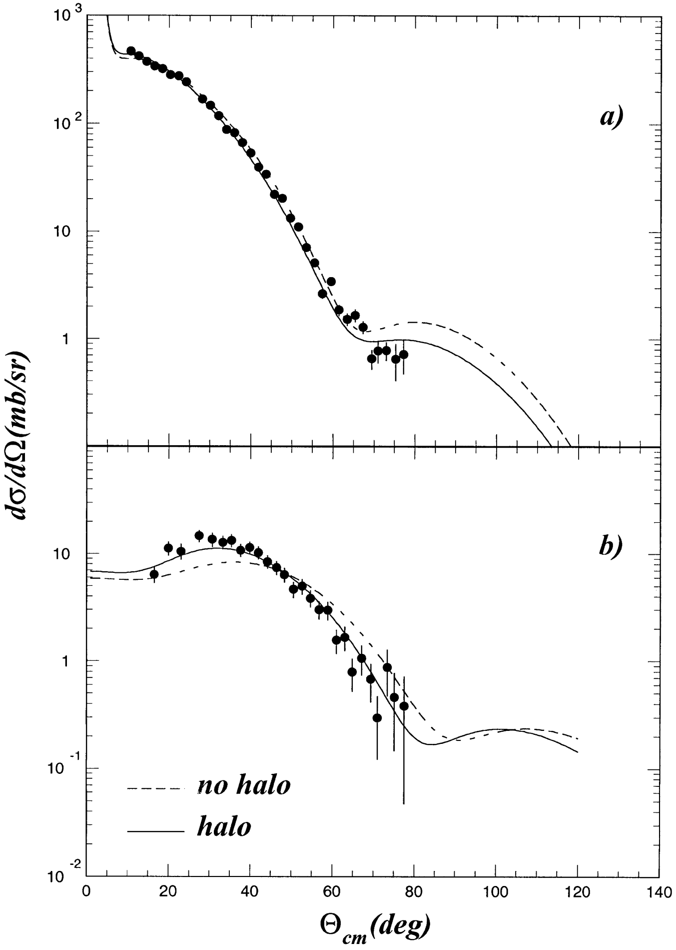
We shall demonstrate the first point in later lectures when we investigate these reactions.

Here, let us return to light ion elastic scattering, specifically proton scattering.

What does proton elastic scattering actually tell us?

The rms matter radius of the target nucleus and not much else, in most cases. Using sophisticated microscopic models to calculate the proton-nucleus potentials it is found that, provided the target density used has the correct rms matter radius, the fit to the elastic scattering data is more or less insensitive to other details of the nuclear matter density.

Take the case of ^{6}He(p,p). The ^{6}He nucleus is what is termed a “halo” nucleus. It has a compact core (in this case a ^{4}He) surrounded by an extended “halo” of neutrons. However, the impact of this halo on the calculated elastic scattering is marginal:



Even if there were data for $\theta_{c.m.} > 70^\circ$ the influence of the halo is similar to (or even smaller than) that of couplings to reaction channels, the effect of which is not included in the formalism.

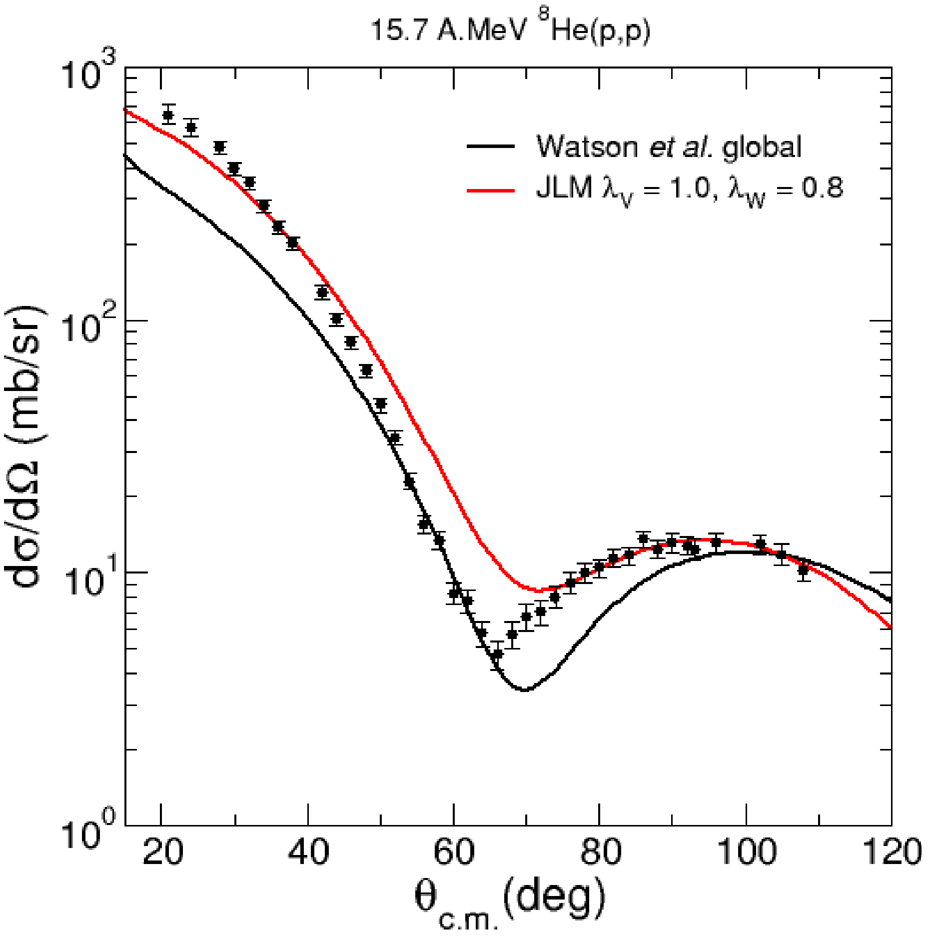
Taken from Lagoyannis *et al.*, Phys. Lett. B **518**, 27 (2001).

So much for what we might term “static” effects due to the specific nuclear structure of the target.

As we said previously though, fitting a large body of proton elastic scattering data with the optical model may be likened to the work of Kepler in reducing Tycho Brahe's mass of planetary observations to his three laws.

It is for targets where the optical model fails (or at least has difficulty) in fitting data with standard forms or where averaged (what we term “global”) OMP parameters describe the data poorly that we should look for “dynamic” effects due to the specific nuclear structure.

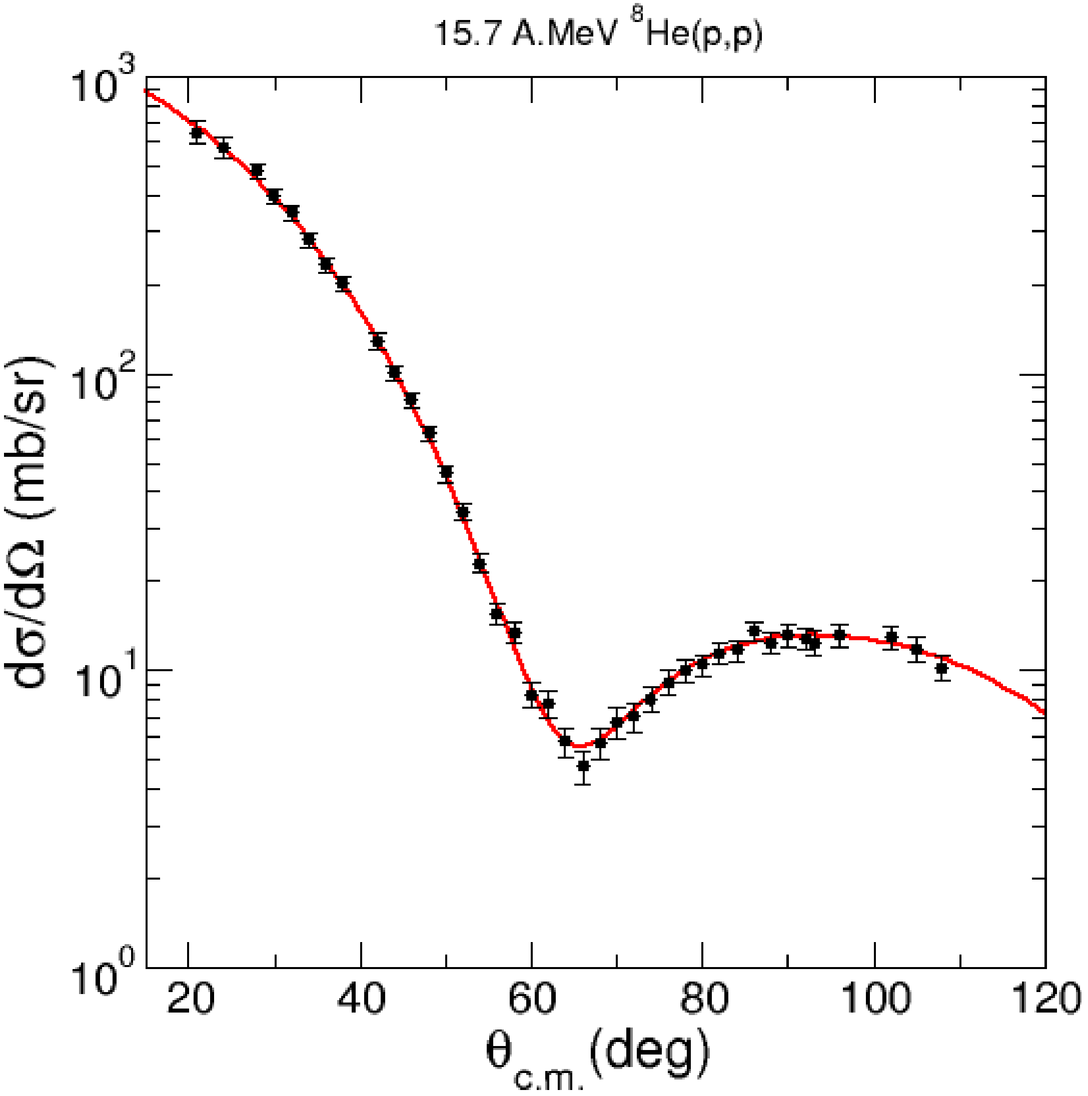
A good example of strong dynamic effects is seen in the 15.7 MeV p + ^{8}He elastic scattering data from GANIL. Neither global parameters adapted to stable p-shell nuclei nor JLM based on a realistic ^{8}He matter density can describe them:



A standard OMP can easily be adjusted to fit these data with reasonable parameters, however:

Data from Phys. Lett. B **619**, 82 (2005)

Watson *et al*. OMP: Phys. Rev. **182**, 977 (1969)



Best fit parameters are:

V = 128.8 MeV, r_V = 1.48 fm, a_V = 0.509 fm
W_d = 19.33 MeV, r_d = 1.173 fm, a_d = 0.465 fm
V_{so} = 5.5 MeV, r_{so} = 1.136 fm, a_{so} = 0.57 fm

$\chi^2/N = 0.32$

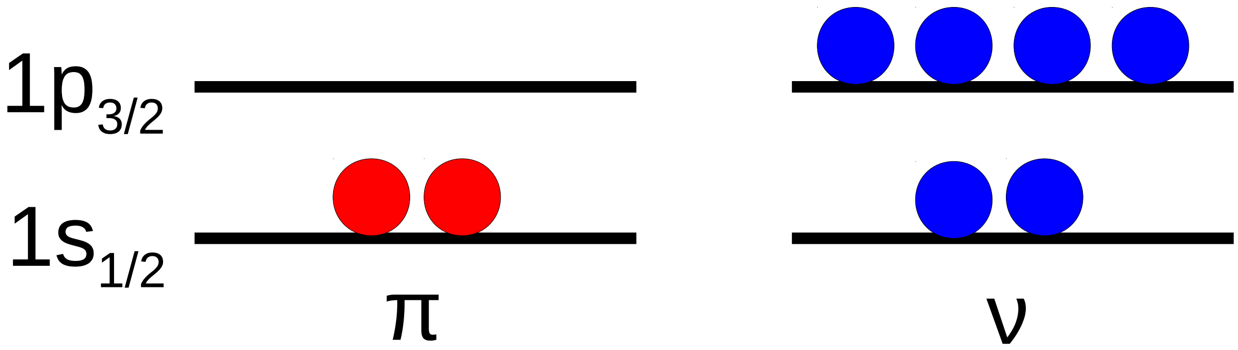
c.f. Watson *et al*. global OMP:

V = 69.73 MeV, r_V = 1.136 fm, a_V = 0.57 fm
W_d = 13.94 MeV, r_d = 1.136 fm, a_d = 0.5 fm
V_{so} = 5.5 MeV, r_{so} = 1.136 fm, a_{so} = 0.57 fm

χ^2/N = 20.4 (34.0 for JLM)

What causes this large deviation from “average” behaviour?

Consider the structure of ^{8}He in terms of the shell model:



The closed ν $p_{3/2}$ sub-shell means that when a proton interacts with a ^{8}He there is a large probability that it will pick up a neutron to form an outgoing deuteron. This feeds back on the elastic scattering, mainly in the form of increased absorption. This will be reflected in the “best fit” OMP parameters.

What of heavy ion elastic scattering? Surely that is uninteresting since it is only sensitive to the OMP in the surface? Also, it is not amenable to obtaining global potential parameters.

In fact, we have already seen that heavy ion elastic scattering can be sensitive to differences in the nuclear structure of the interacting nuclei under the right conditions:

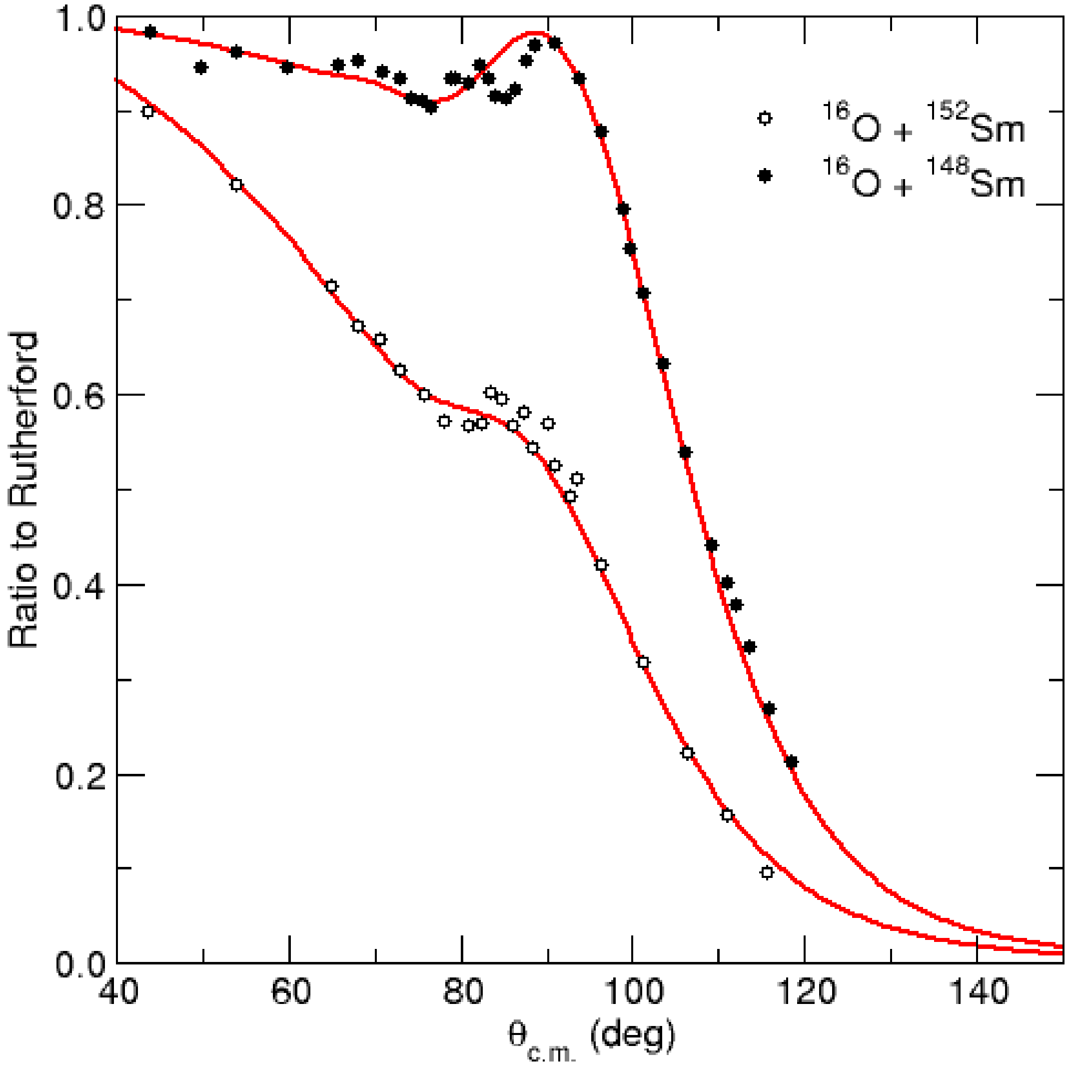
1) Energy range – close to the Coulomb barrier
2) Precise measurement – ~ 2-3 % uncertainty
3) Careful choice of target and/or projectile

A sub-set of heavy ion elastic scattering data that are particularly sensitive to nuclear structure are those systems where there is strong coupling to other reaction channels, usually inelastic excitations.

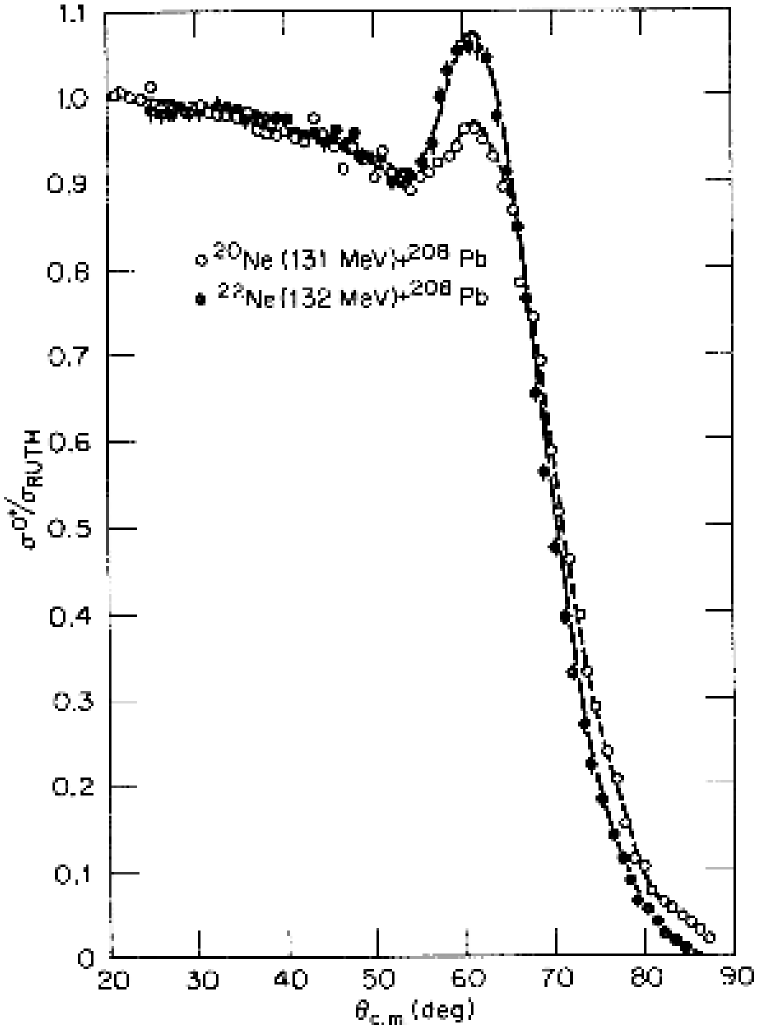
Under the right conditions such data can be *quantitatively* sensitive to the nuclear structure, although they are usually difficult, if not impossible, to fit with a conventional OMP.

A recent review article has dealt with this subject: Keeley, Kemper and Rusek, Eur. Phys. J. A **50**, 145 (2014) so I will only show a few examples here.

Two examples, target (left) and projectile (right) coupling:



72 MeV ^{16}O, curves include $0^+ \leftrightarrow 2^+ \leftrightarrow 4^+$ coupling in Sm; B(E2) for ^{152}Sm nearly X 5 that for ^{148}Sm (E_{ex} also lower than 148)



Taken from Phys. Rev. C **29**, 459 (1984)

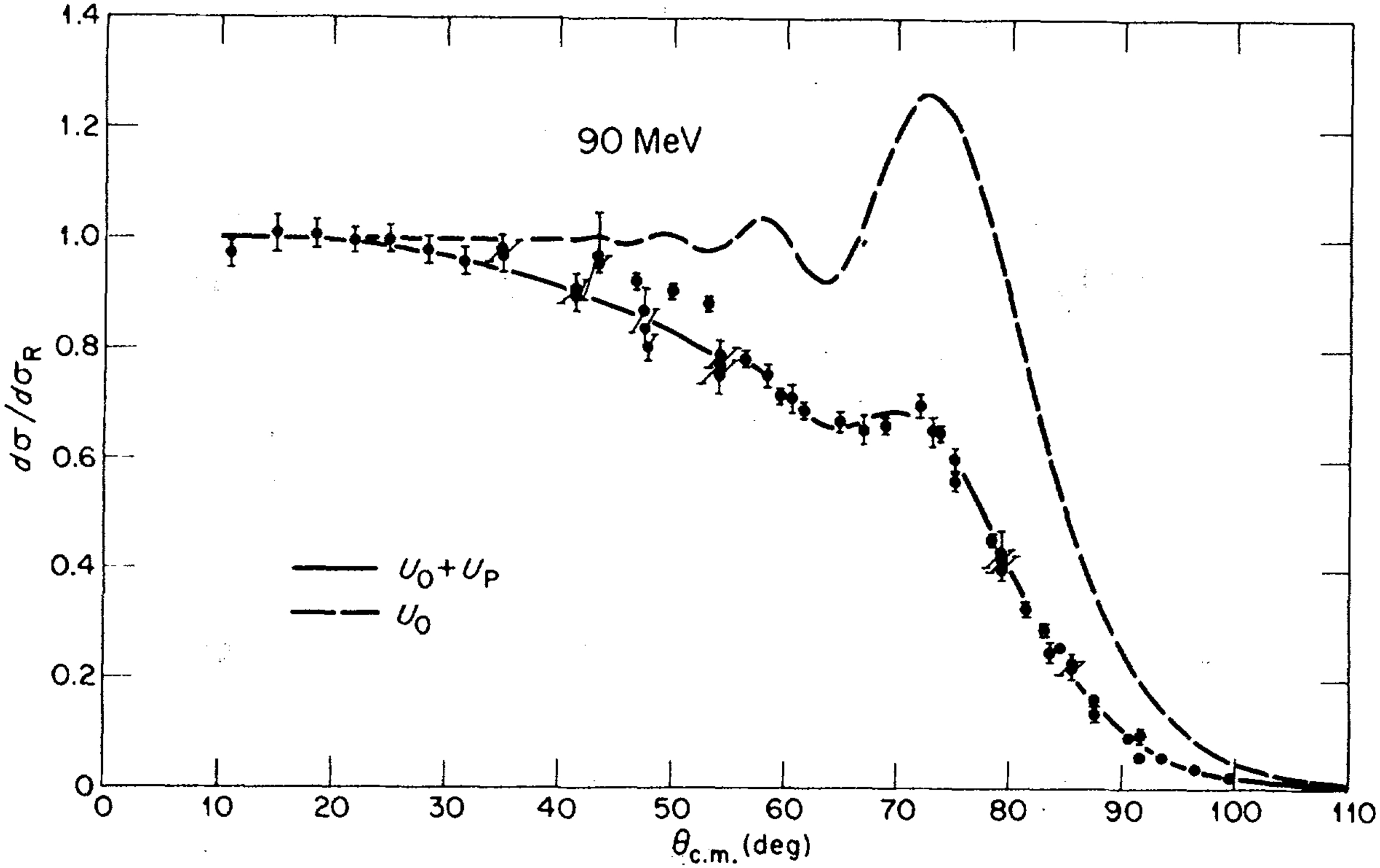
These are examples of strong coupling to *inelastic excitations*.

The elastic scattering is sensitive to the B(E2) value of the relevant coupling almost exclusively (it is the Coulomb excitation that is important here since the charge product Z_1Z_2 is large).

Such data are difficult to fit with an OMP of standard form. One can obtain very good descriptions either by explicitly including the coupling (coupled channels calculations,as in the previous slides, which we shall consider in the next lecture) or by adding a long-range imaginary DPP to the usual OMP.

The form of the latter can be estimated using semi-classical theories of Coulomb excitation and also works very well …

If appropriate approximations are made an analytic expression for a long-range imaginary DPP is obtained (the real part is negligible under these conditions) which, when added to a conventional OMP, describes the elastic scattering well:

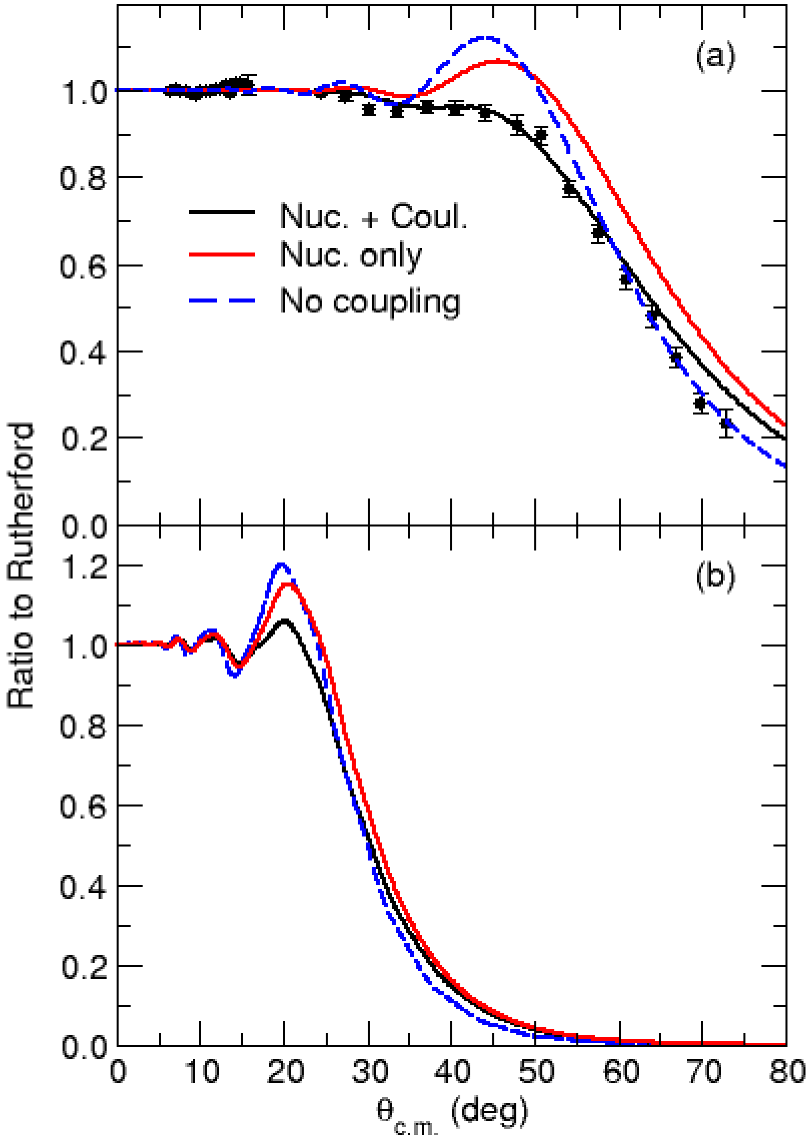


Taken from Love, Terasawa and Satchler, Phys. Rev. Lett. **39**, 6 (1977).

Apart from the OMP parameters (obtained in this case by fitting the *quasi-elastic scattering* AD) the only parameter in this model is the B(E2) value.

The only problem with this type of study is that it is essential to measure the pure elastic scattering. Since the strongly coupled states tend to have small excitation energies (~100 keV or even less) this can be a major experimental headache! Often magnetic spectrometers are required to obtain the necessary resolution, which leads to a long experiment.

Similar effects occur in the elastic scattering of many light radioactive beams, the classic example being ^{6}He scattered from a ^{208}Pb target:



(a) 27 MeV ^{6}He + ^{208}Pb
Kakuee *et al.*, Nucl. Phys. A **728**, 339 (2003)

(b) 45 MeV ^{6}He + ^{208}Pb

Two things to note:

1) Effect is still dominated by Coulomb excitation (Coulomb breakup here)

2) The effect diminishes as the incident energy is increased above the barrier

For these nuclei, measuring pure elastic is not a problem (no bound excited states!)

There are many other examples of radioactive beams where this behaviour is known or expected. There is still plenty of work to be done in this line.

I hope that this small digression has shown that elastic scattering is interesting for its own sake and not just as an ingredient to reaction studies.

Fitting data with an optical model may seem to be a somewhat pointless exercise in its own right, but such fits can give us clues to important physical effects that require more sophisticated models to describe.

Attempting to systematise large bodies of data through the search for simple laws that govern their behaviour is a first step to true understanding: without Kepler's work Newton would have had a much harder task in formulating his theory of gravitation.

We might not yet have our Newton for elastic scattering but we certainly still need our Keplers!

Lecture 7: inelastic scattering

To recap, inelastic scattering takes place when we have an inelastic collision between two interacting nuclei; i.e., one (or both for heavy ion projectiles) of the nuclei involved is raised to an excited state during the collision with a corresponding loss in the total kinetic energy of the system.

The excitation usually takes one of two basic forms: single particle (or hole) excitations or collective excitations, subdivided into vibrational and rotational excitations.

This is, of course, a simplified picture, and some excitations do not fall neatly into a particular category.

We shall not consider single particle excitations here and concentrate on collective – vibrational and rotational – excitations.

As the name implies, these are collective motions of the nucleus as a whole and can be pictured by analogy with the motions of a drop of liquid.

If this drop is deformed, i.e. non-spherical, in shape it can rotate about an axis of symmetry to produce rotational excitations. If spherical, it can vibrate in various ways to produce vibrational excitations.

Deformed nuclei can also vibrate, so they usually have a mixture of rotational states (divided into *bands* of related states) and vibrational states (which may be the starting point – the *band head* – of a rotational band).

We will not go further into the details of nuclear structure here – it is a specialist subject in its own right – but will concentrate on how we model inelastic excitations using direct reaction theory.

The theory is identical for light and heavy ions, but Coulomb excitation is obviously much more important for heavy ion inelastic scattering.

To begin, we take the case of an even-even rotational nucleus. How do we model the excitation of the first member of the ground state rotational band, i.e. the $0^+ \rightarrow 2^+$ transition?

We need to calculate the matrix elements for the set of coupled equations (recall lecture 2). To do this we need an interaction potential that can take account of the collective excitations.

This is most simply done with a deformed optical potential. If we assume that the nucleus is permanently deformed then the radius is:

$$R(\theta, \phi) = R_0 \left\{ 1 + \beta Y_2^0(\theta, \phi) \right\}$$

β is the deformation parameter. If we now assume that the potential depends on the distance from the nuclear surface [e.g. in a Woods-Saxon potential we replace *R* by *R*(*θ*,*φ*)] and expand to first order in β we obtain:

$$V\left\{r - R(\theta,\phi)\right\} = V(r - R_0) - \beta R_0 Y_2^0(\theta,\phi)\frac{\mathrm{d}V}{\mathrm{d}r}$$

The same formalism may be applied to the excitation of a 2^+ single phonon state in a vibrational nucleus only now the radius is given by:

$$R(\theta,\phi) = R_0\left\{1 + \sum_m \alpha_m^* Y_2^{m*}(\theta,\phi)\right\}$$

and, again to first order in β, the potential is now:

$$V\left(r-R\right)=V(r-R_0)-\sum_{m}\alpha_m^* Y_2^{m*}(\theta,\phi)R_0\frac{\mathrm{d}V}{\mathrm{d}r}$$

and we may define a root-mean-square deformation parameter as:

$$\beta^2=\left\langle 0\mid\sum_{m}\mid\alpha_m\mid^2\mid 0\right\rangle$$

These transition potentials may be used to calculate the matrix elements in either DWBA or CC formalisms

A similar exercise may be performed for the Coulomb excitation.

In DWBA there is no difference between excitation of a single quadrupole phonon in a vibrational nucleus and excitation of the first 2^+ state of the ground state band of a rotational nucleus.

However, if we use the more accurate CC theory, second order effects come into play, and for the rotational nucleus *reorientation* terms appear. These are diagonal matrix elements linking states in different magnetic sub-states. They do not exist in vibrational nuclei since these matrix elements are then zero.

In principle, CC should be used for collective transitions since they are strongly coupled, but DWBA is still sometimes used, particularly for the special case of giant resonances.

However, there are several important considerations that are common to both formalisms as applied to inelastic scattering.

We have seen that in addition to the spherical part of the optical potential we now need to define a deformation parameter β. In fact, this enters the equations as the combination βR_0, referred to as the *deformation length*, δ, usually given in fm.

Thus, in principle, we need only define β in addition to the usual optical potential to describe inelastic scattering. However, there are some choices to be made!

1) There are nuclear and Coulomb potentials. Should $\beta_{Coul} = \beta_{Nuc}$? Or should $\delta_{Coul} = \delta_{Nuc}$? Or are the two independent?

2) The nuclear potential has both real and imaginary parts. Should $\beta_{real} = \beta_{imag}$ or $\delta_{real} = \delta_{imag}$? Or are the two independent?

3) Should we deform the imaginary part at all?

To answer these questions, it should first be stressed that both β and δ are model dependent, i.e. the exact value extracted by fitting a given data set will depend on the details of the model used. However, generally δ is better defined than β and I personally prefer to work with it (most reaction codes use β, although FRESCO uses δ).

In principle, one expects that the Coulomb and nuclear responses should be the same in the collective model, so we would expect $\delta_{Coul} = \delta_{Nuc}$. However, remember that our “nuclear fluid” has two components, neutrons and protons …

The Coulomb response depends on the distribution of the protons alone, whereas the nuclear response will also depend on the distribution of the neutrons. They need not coincide exactly, and for more exotic nuclei there is no *a priori* reason why $\delta_{\mathrm{Coul}} = \delta_{\mathrm{Nuc}}$.

However, the nuclear response ought to be the same for both real and imaginary parts. Since R_0 will not in general be the same for both parts I prefer to fix $\delta_{\mathrm{real}} = \delta_{\mathrm{imag}}$. It is, however, a moot point whether one should deform the imaginary part of the potential at all; certainly for heavy ions fits to data are worse if this is not done.

We can at least avoid the question of what to take for β_{Coul} (or δ_{Coul}) by using the B(Eλ) value instead for the Coulomb coupling.

B(Eλ) is the electric transition probability for the 2^{λ}-pole transition and is usually expressed in units of e^2b^{λ}, where e is the charge on the electron. N.B. in reaction work we use B(Eλ)↑, unlike in gamma spectroscopy where B(Eλ)↓ is quoted.

FRESCO, for example, uses the reduced matrix elements for Coulomb excitation, related to the B(Eλ). Other codes use β_{λ} for the Coulomb as well as the nuclear excitation.

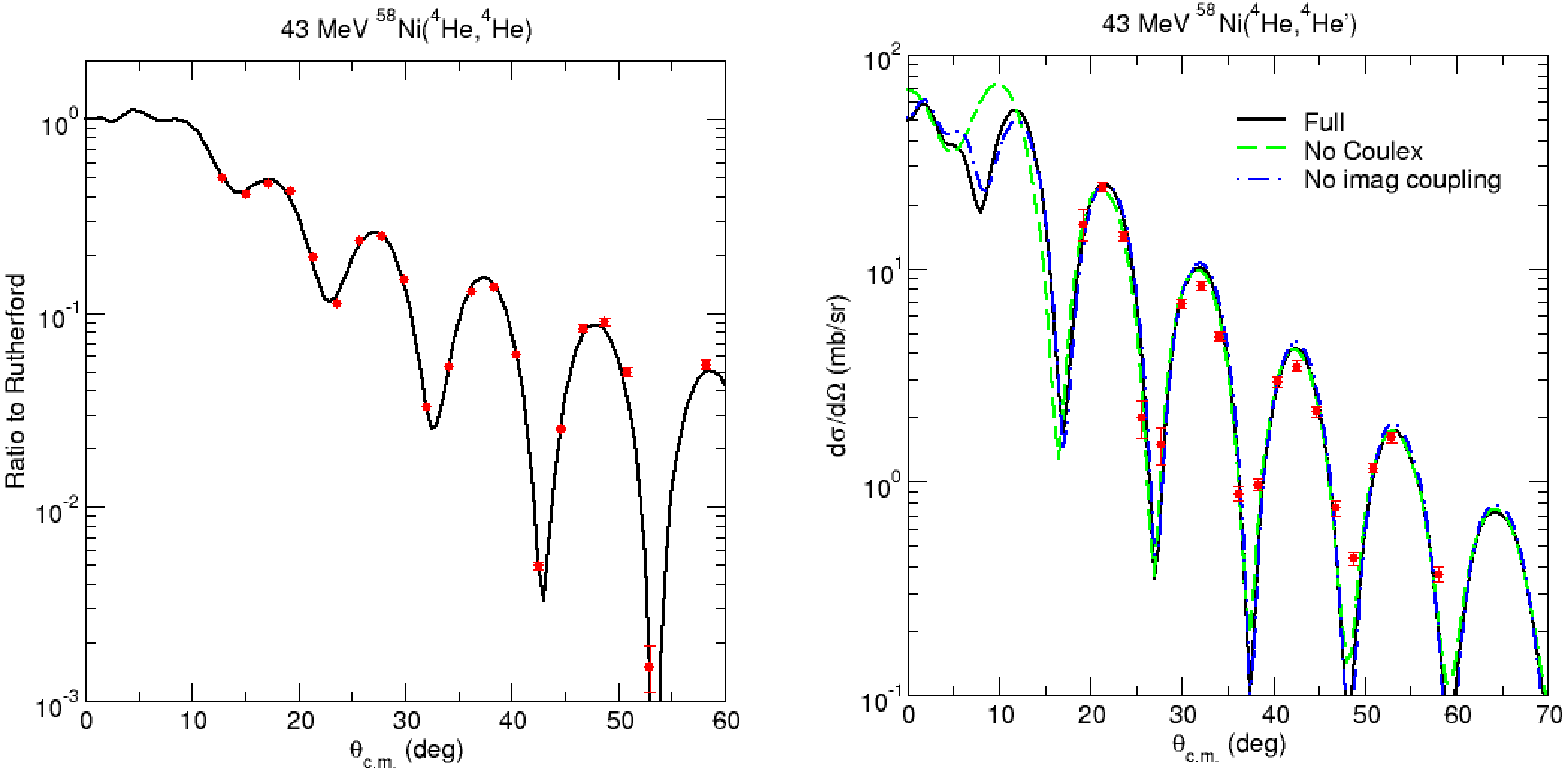
The advantage of using B(Eλ) is that it can be independently determined (e.g. by electron scattering or Coulomb excitation) and is thus independent of the charge radius chosen. We can relate B(Eλ) to β_λ within the collective model as follows:

$$B(E\lambda) = (3/4\pi)^2\ (ZeR^\lambda)^2\beta_\lambda{}^2$$

where R is the nuclear charge radius. Thus, the value of β_λ obtained depends on the choice of R.

How does all this work out in practice? Let us take a few typical examples of inelastic excitation with both light and heavy ions.

We take as our first example 43 MeV ^{58}Ni(α,α’) to the 1.45 MeV 2^+ state of ^{58}Ni. We begin with DWBA:



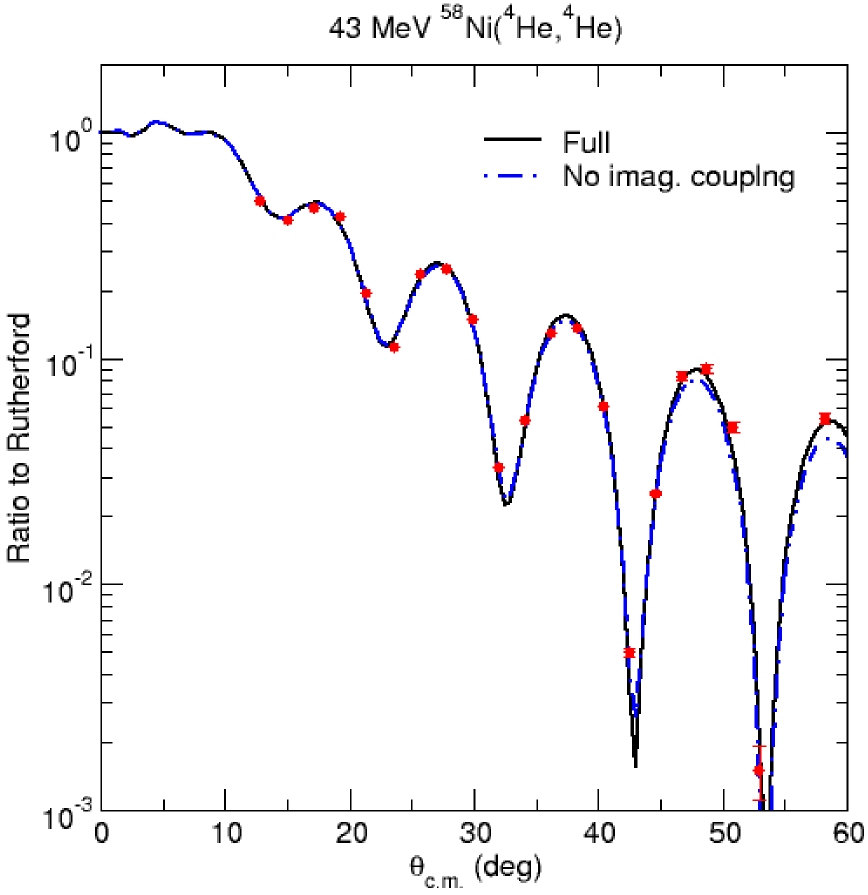
We fix the B(E2) at the known value and adjust δ_2 to fit the inelastic data

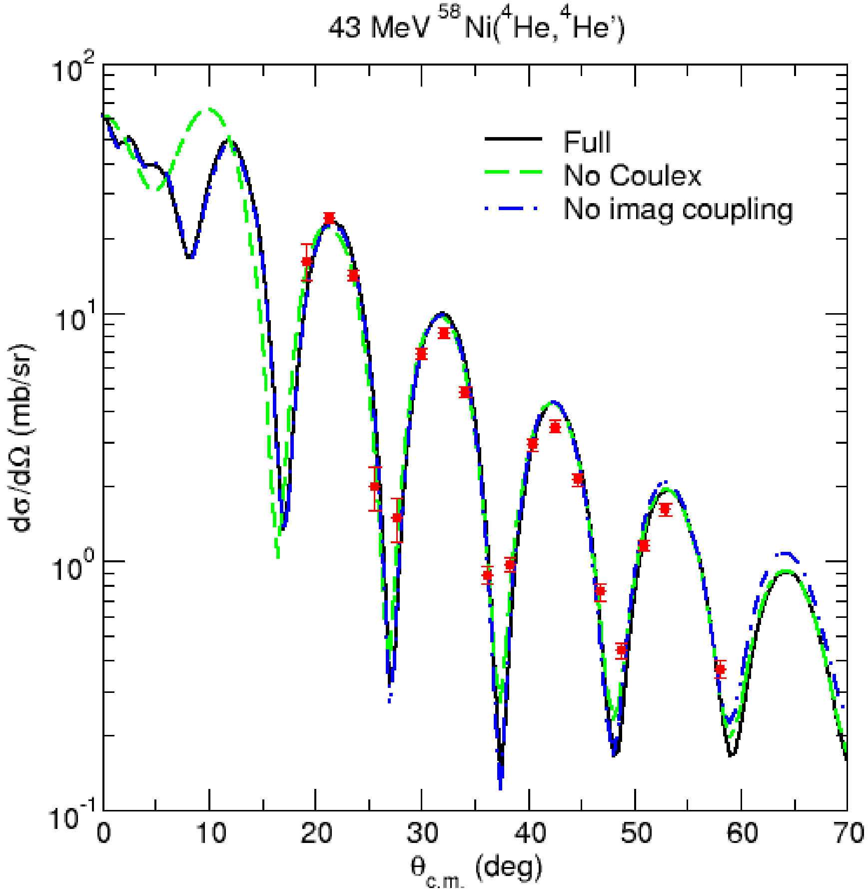
The description is good, we obtain δ_2 = 1.10 fm.

Note that we are insensitive to the inclusion of an imaginary coupling potential in this case, and only sensitive to the inclusion of Coulomb excitation at forward angles (where there are no data).

Thus, for inelastic scattering of α particles from a medium mass vibrational target nucleus DWBA seems to work well. What do we find if we perform a coupled channels analysis?

Coupled channels analysis of the same data. We fix B(E2) at the same value as before but this time we have to search on the OMP parameters as well as δ_2 since in CC calculations the “back coupling” affects the elastic scattering.





The result is very similar to DWBA, although we have had to readjust the OMP parameters. We again see little sensitivity to the inclusion of either Coulomb excitation or imaginary coupling potentials (the omission of the latter does have a slight effect on the elastic scattering).

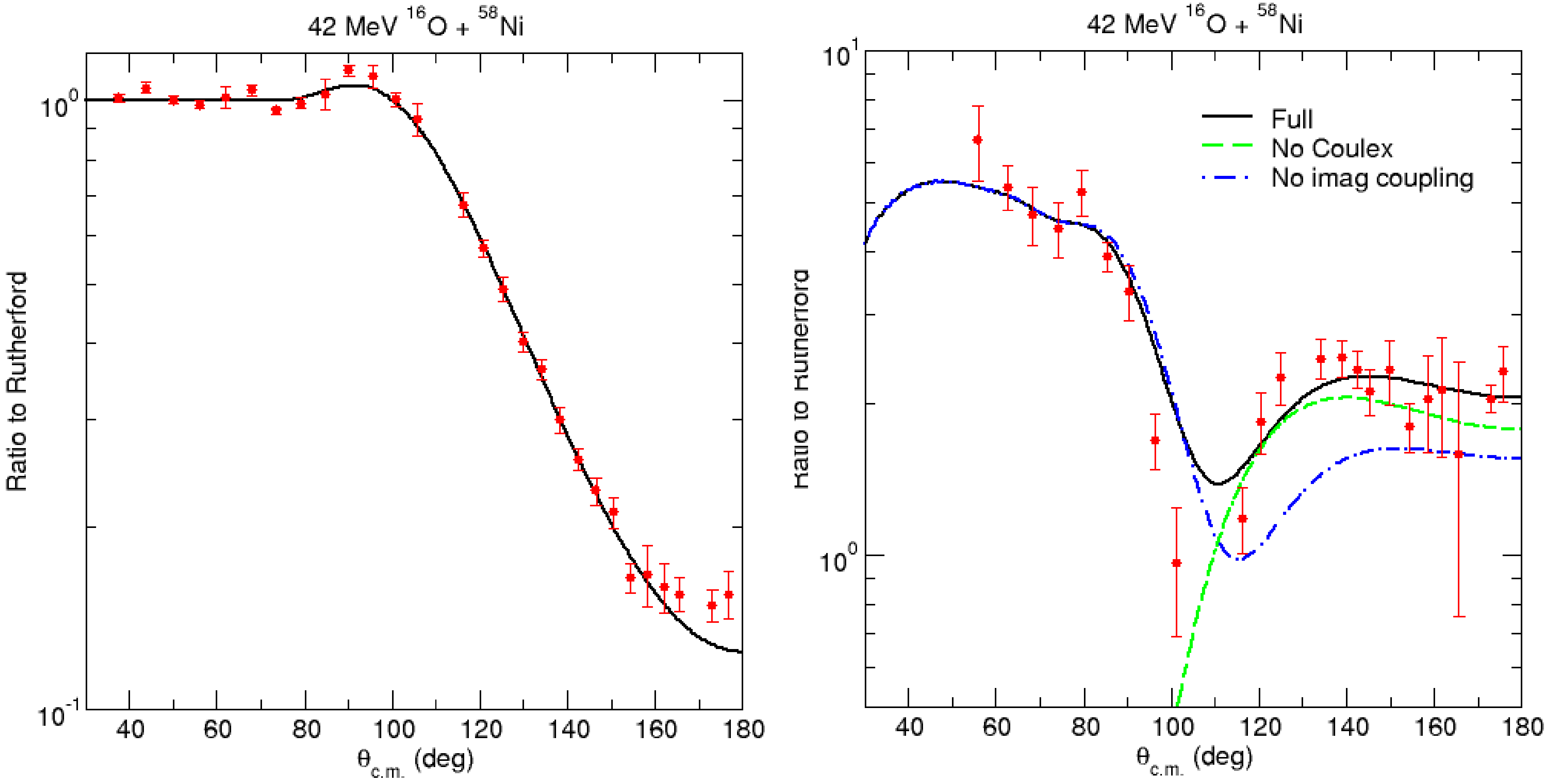
The value of δ_2 obtained is 10 % smaller than in DWBA, δ_2 = 1.00 fm, within the likely uncertainty.

Thus in this case DWBA seems perfectly adequate. This is largely due to the relatively weak coupling strength.

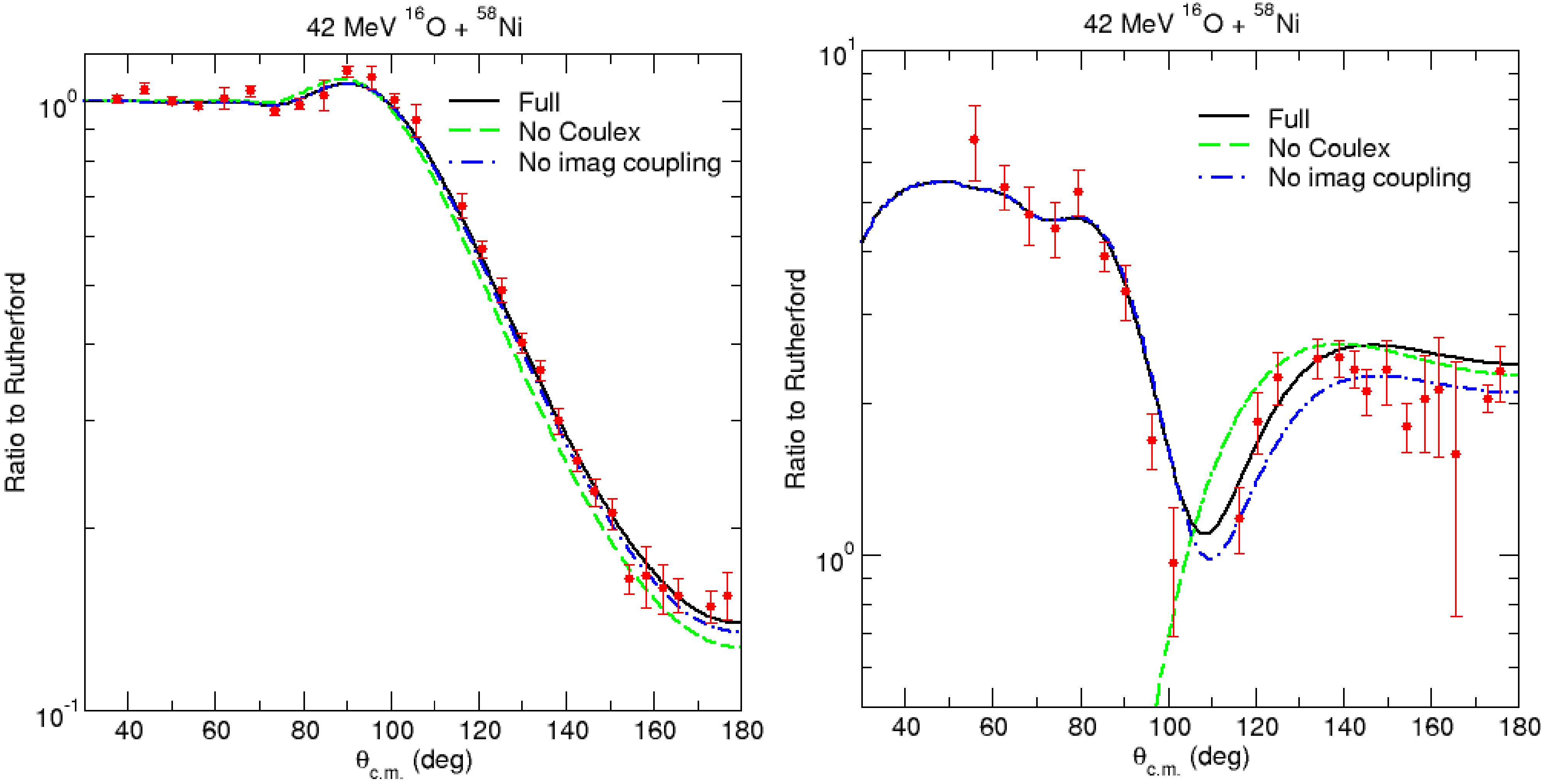
As an example of a vibrational coupling in heavy ion inelastic scattering, we stick with a ^{58}Ni target but this time excite it with a ^{16}O beam. We again perform both DWBA and CC analyses and assess the importance of Coulomb excitation and the imaginary coupling.

We take as our data set elastic and inelastic scattering (to the ^{58}Ni 1.45 MeV 2^+ state) at ^{16}O incident energy of 42 MeV. We again include just the $0^+ \rightarrow 2^+$ coupling in the ^{58}Ni target, as in the α scattering, fixing the B(E2) at the same value.

DWBA first. Best fit δ = 1.0 fm, so similar to α scattering result. Note that both Coulomb excitation and imaginary coupling are much more important than for α scattering.



Now for CC. Best fit δ = 1.0 fm, so same as for DWBA, although we have only searched on *V* and *W* so this value could change slightly if we optimised the potential “geometry”.

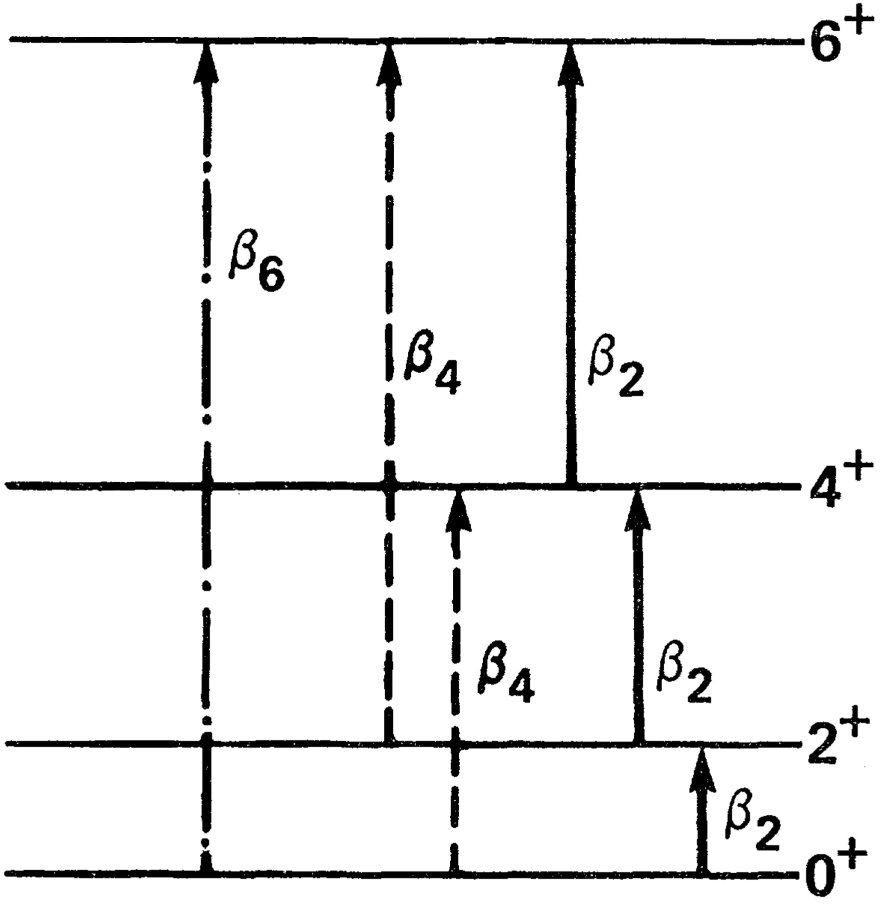


Thus, for heavy ions we also find that DWBA is adequate for a typical vibrational nucleus, although CC does give more consistent results for data sets at different incident energies.

Coulomb excitation is important (as expected) but we also find that imaginary couplings have a significant influence on the inelastic scattering, although this is smaller for CC than for DWBA.

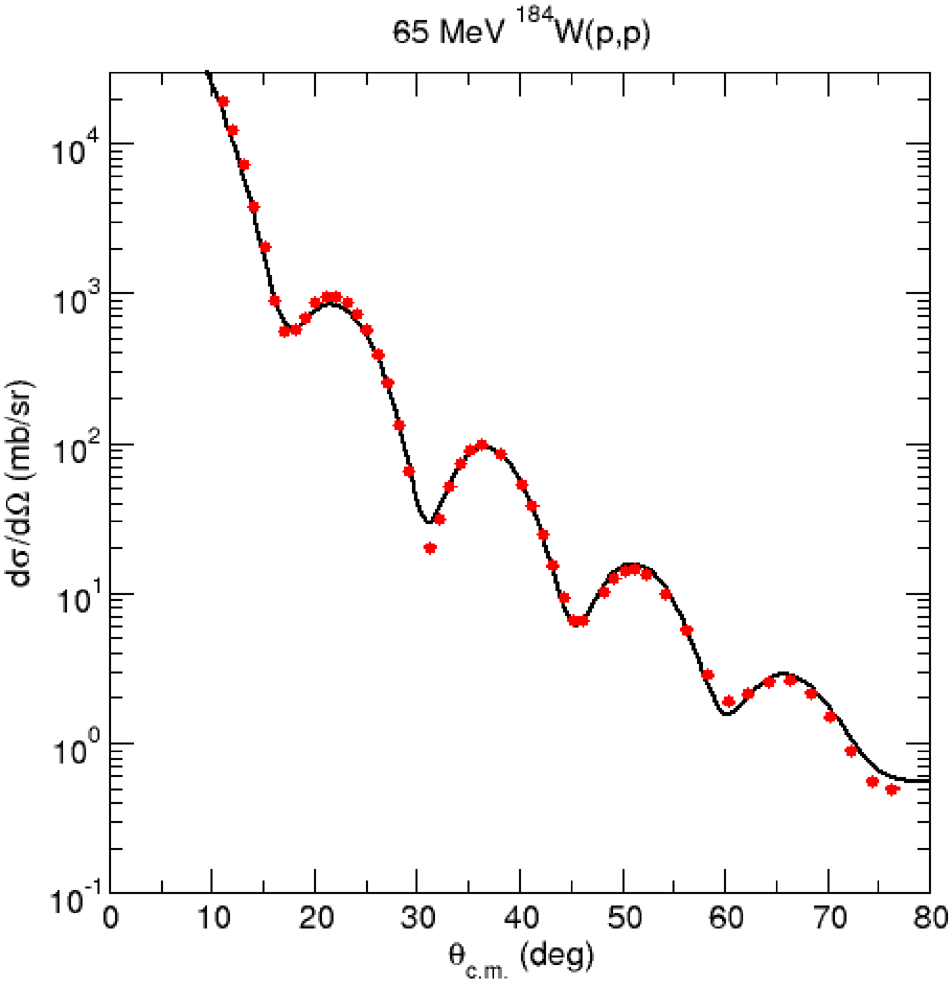
How well does the simple collective model account for couplings to a rotational band?

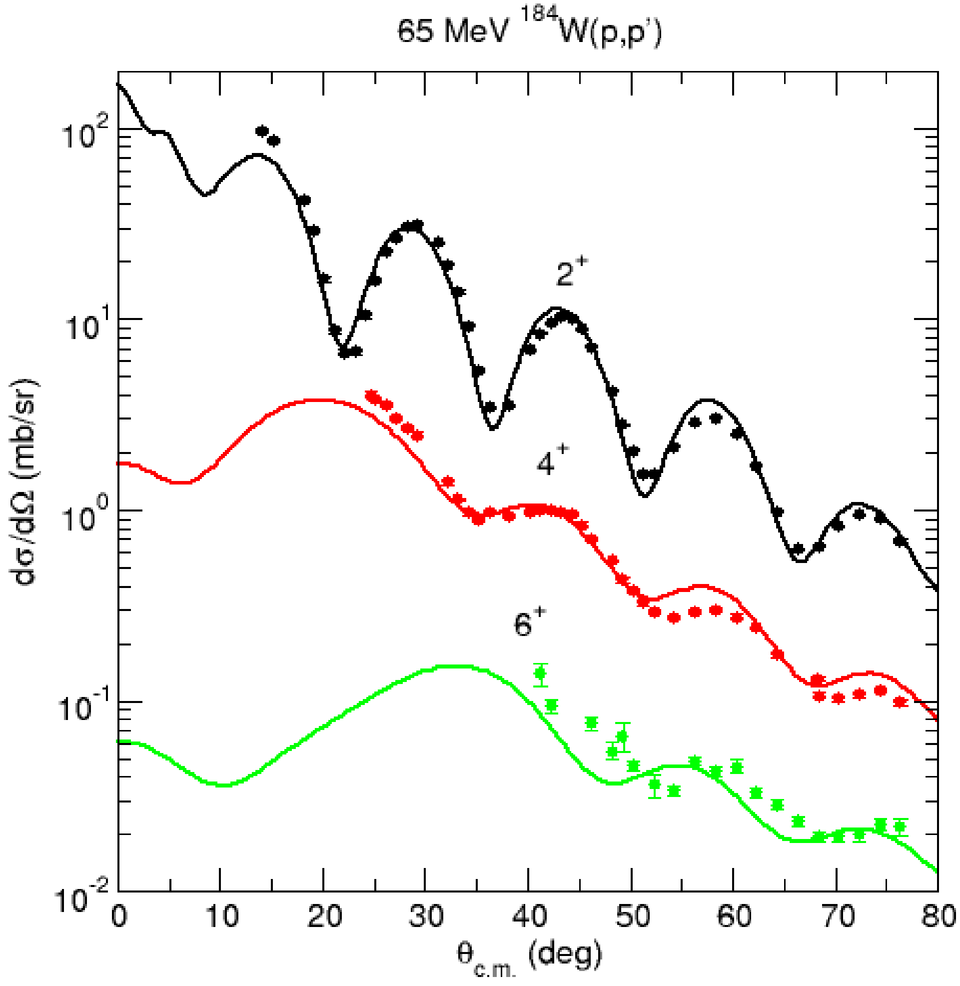
For a rotational band in an even-even nucleus we have the following sequence (for, e.g., the ground state band) of levels: 0^+, 2^+, 4^+, 6^+ etc. If we truncate the series at $J^\pi = 6^+$ the following couplings are possible if we allow up to 2^6 pole coupling:



Due to the multi-step couplings it is not really sensible to use DWBA for such a coupling scheme. We go straight to CC, and in principle have to determine three parameters: β_2, β_4 and β_6.

CC calculations for 65 MeV protons incident on ^{184}W. We assume a pure rotational model for excitation of the first 2^+, 4^+ and 6^+ levels:





Agreement is reasonable. Calculations do not include 2^6 pole couplings: evidence for these in this case is marginal (at least with the available data). At least we can say that any 2^6 pole coupling strength is small in this model.

We have thus seen that the pure rotational model also works rather well in a CC calculation.

In the next lecture we look at some refinements in how the calculations are performed and investigate semi-microscopic methods.

Lecture 8: inelastic scattering continued

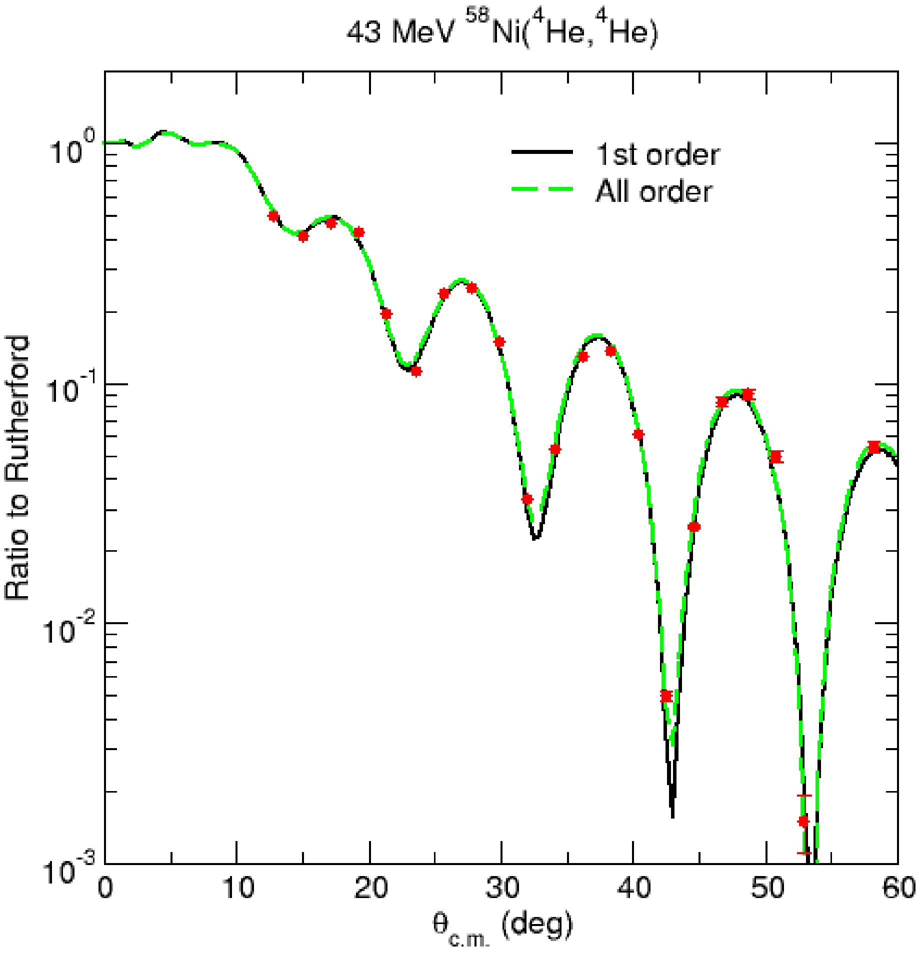
In the previous lecture we saw how DWBA or CC calculations using the collective model were able to describe rather well data for systems with collective motion.

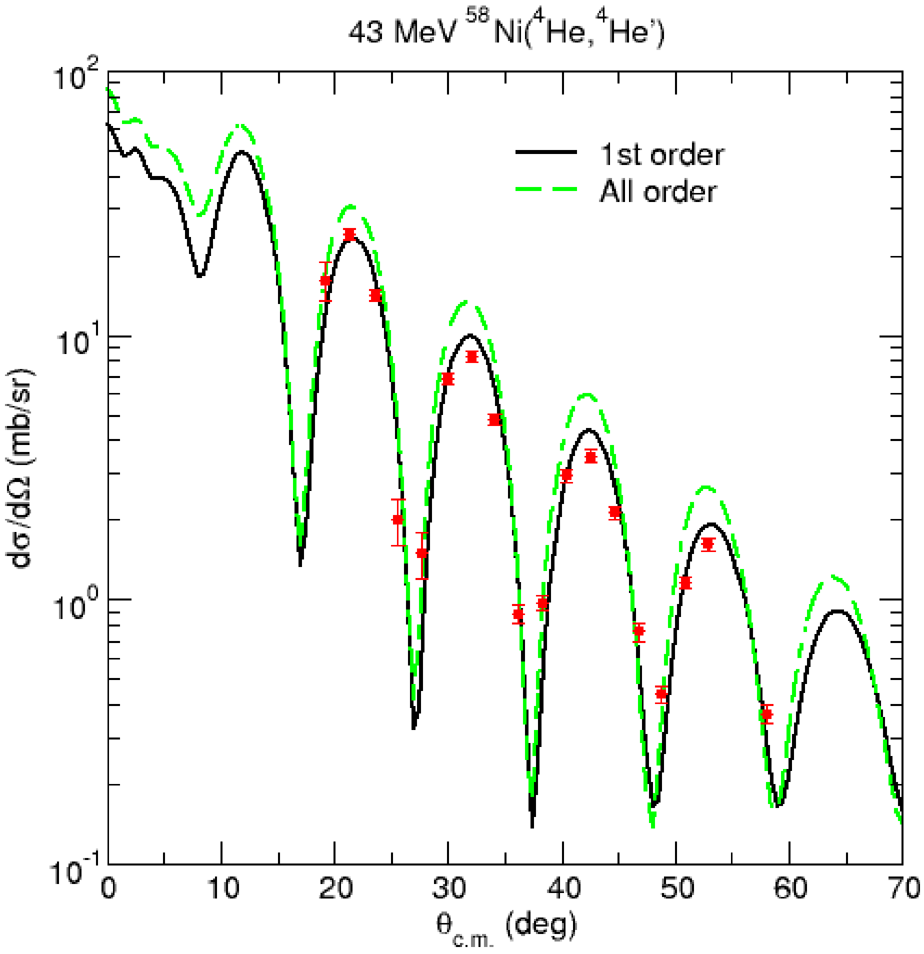
However, you will recall that we simplified the treatment by only taking the expansion of the deformed radius to first order in β. This has been standard practice for many years, but is known to be a relatively poor approximation if β is large.

Since DWBA is first order anyway refinements only make sense for CC calculations.

Many codes now allow treatment of the deformation to all orders (e.g. ECIS, FRESCO, CCFULL).

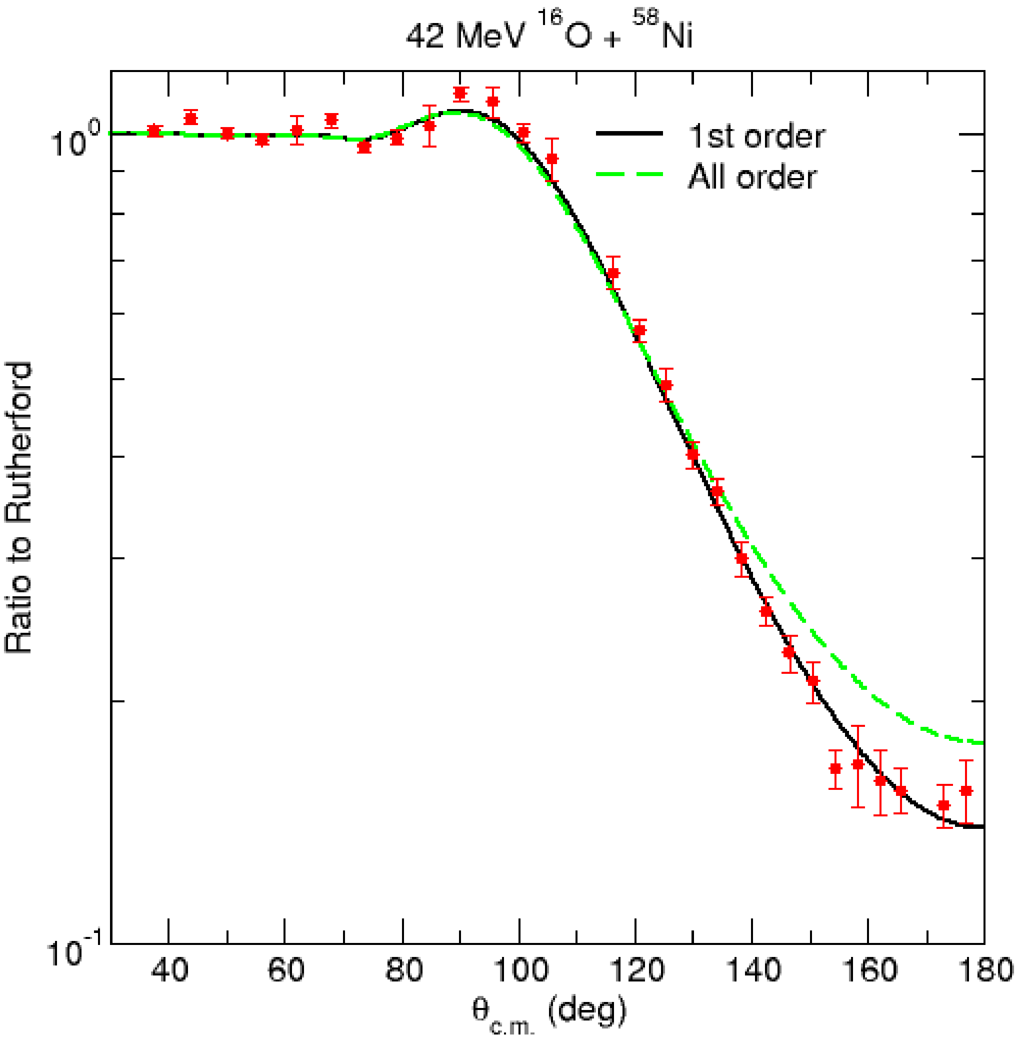
What difference does this make to our examples?

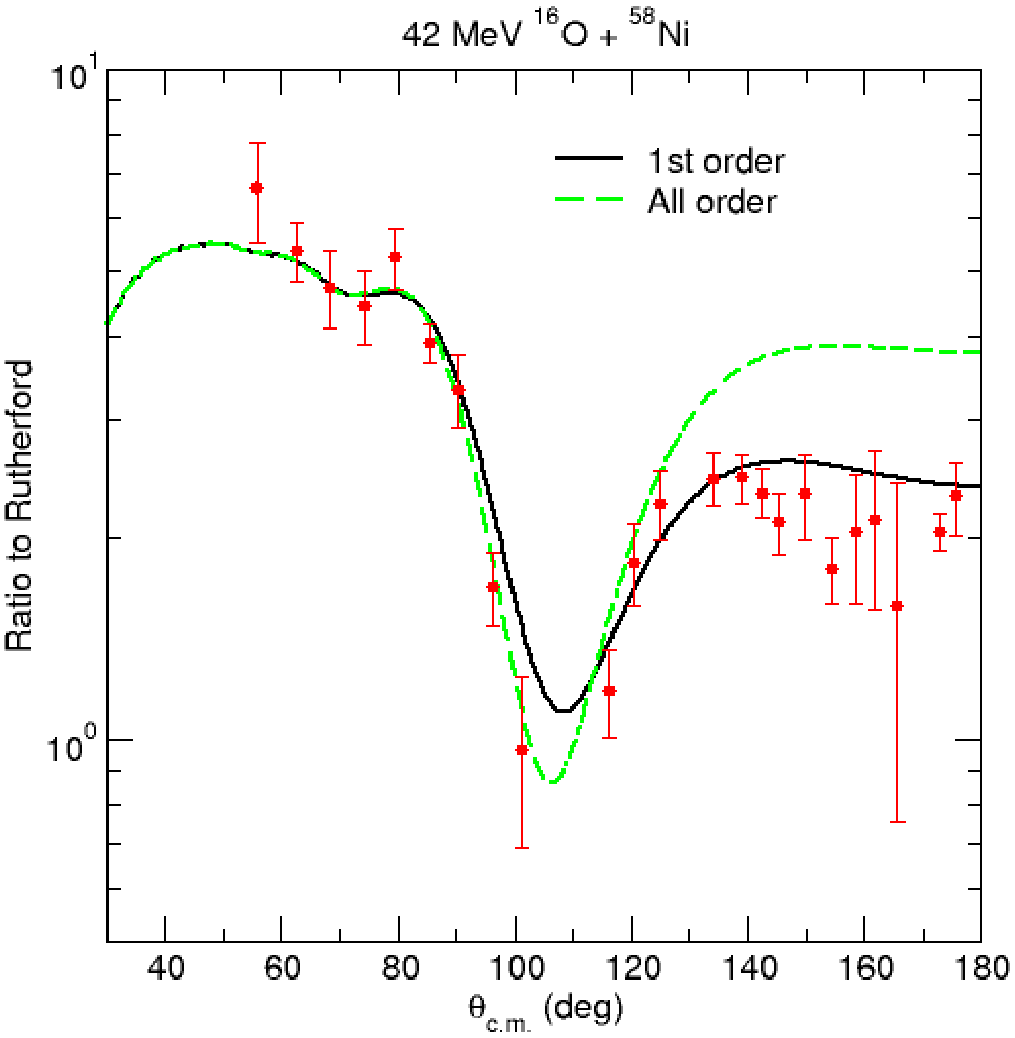




Amounts to an ≈ 15 % reduction in δ

And for a heavy ion projectile?



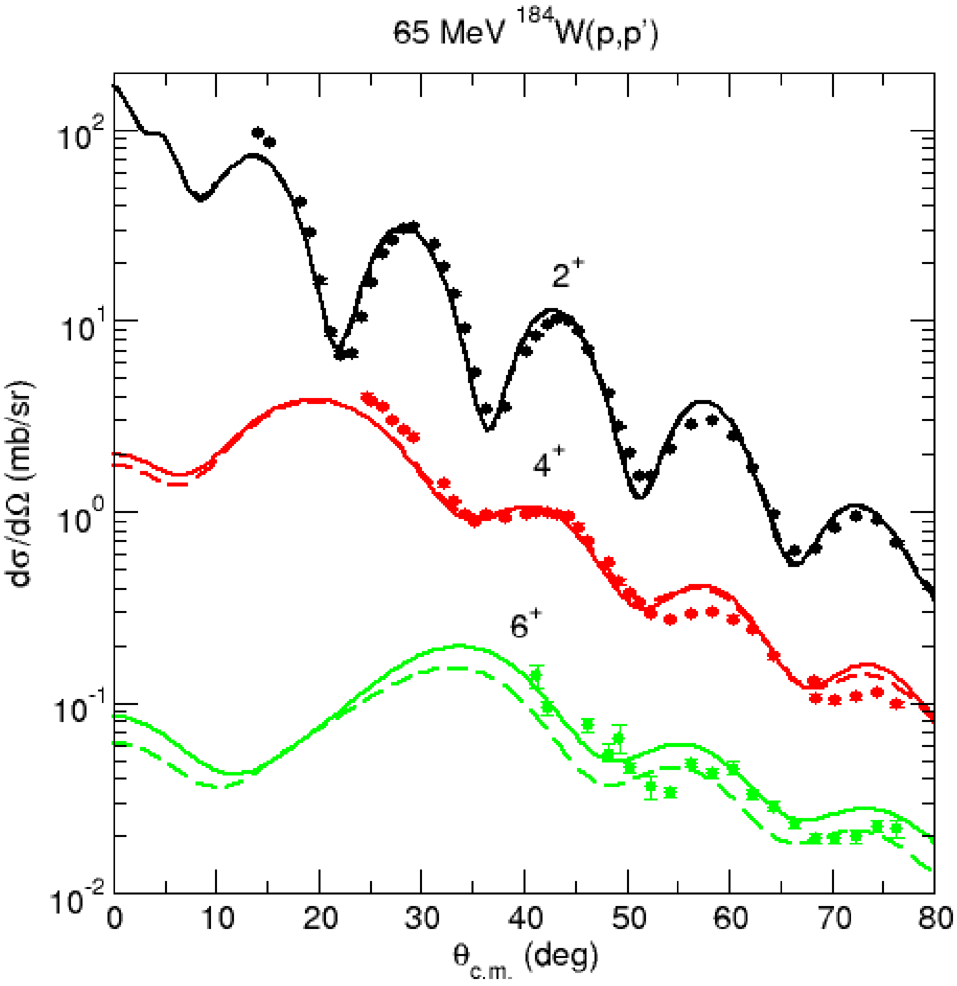


Amounts to an ≈ 20 % reduction in δ

We find that even for a relatively weak vibrational coupling (the $0^+ \rightarrow 2^+$ coupling in ^{58}Ni) there is a significant change in the value of the deformation extracted if we go to all order, even if the shapes of the angular distributions are not significantly altered.

What happens in our rotational coupling case where the coupling strengths are stronger?

There is some slight improvement in the shape of the 4^+ and 6^+ angular distributions (the elastic scattering is unchanged) but while δ_2 is unaltered, δ_4 has increased in magnitude by ~ 42 %.



Dashed curves denote previous 1st order result and solid curves the new all order one.

Due to the more complex coupling scheme it is not obvious whether the lack of change in δ_2 is significant

Since the question of all order versus first order only affects the nuclear excitation (it does not apply to the Coulomb excitation part) it will be seen that drawing quantitative conclusions from deformation parameters (either β or δ) extracted from fits to inelastic scattering data can be problematic.

This is particularly so if we want to look for evidence of $\beta_{\text{Coul}} \neq \beta_{\text{Nuc}}$.

We have also confined ourselves to the strict collective model (at least for the rotational case) in that within a given band β_λ values (including reorientation couplings) are the same for all transitions.

This need not be the case, and in general it is not true (although it is often a good approximation). If the information is available (e.g. static quadrupole moments of non spin-zero states for reorientation couplings) most codes allow the strength of each coupling to be set individually.

One could of course try to determine the individual strengths by fitting a set of inelastic scattering data. This brings us back to the old problem of trying to fit more parameters to a limited data set, but there can be cases where it is impossible adequately to “fit the elephant” otherwise!

Ultimately, how serious this model dependence of the nuclear deformation parameters is will depend on the use one wishes to make of the inelastic scattering data. If one is attempting to draw nuclear structure conclusions from β or δ values extracted from inelastic scattering data it can be a significant problem.

On the contrary, if one is mainly interested in the coupling effect of inelastic excitations on the elastic scattering (often the case in heavy ion work) then it is only a problem if one is constrained to take β or δ from analyses in the literature, when care should be taken to use the same model.

It would, of course, be more satisfying if we could in some way calculate the transition strengths using a nuclear structure model (which might in principle be more sophisticated than the simple collective model).

Obviously, in the light of what we have just said, calculating the deformation parameters alone would not help us much. Is there another way?

Yes: we may go back to the folding models we discussed for elastic scattering (JLM for nucleons or double-folding for heavy ions). We can equally well use these models with a *transition density* replacing one of the ground state matter densities.

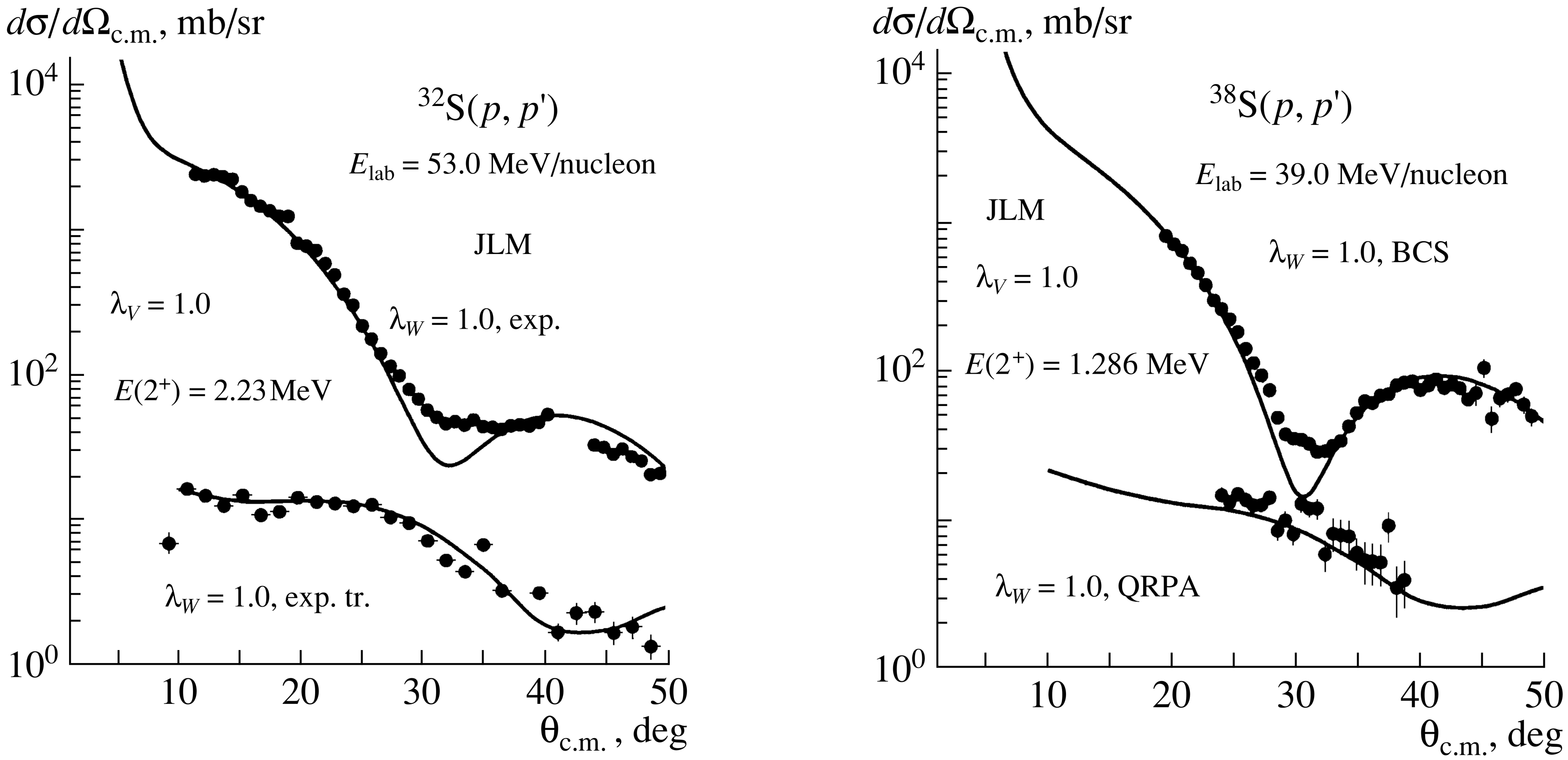
These transition densities may be calculated using a structure model (just as the ground state densities could) and used to calculate the transition potentials required in our inelastic excitation calculations.

Alternatively, we may take a semi-empirical approach and deform the appropriate ground state *density* (as opposed to the OMP). This is known as the *Tassie model*.

Both approaches also get round a problem that may have occurred to you: all* the transition potentials in the “standard” collective model have the same radial shape. There is no reason why this should be so in reality.

* Almost all. Monopole and dipole transitions have a different form

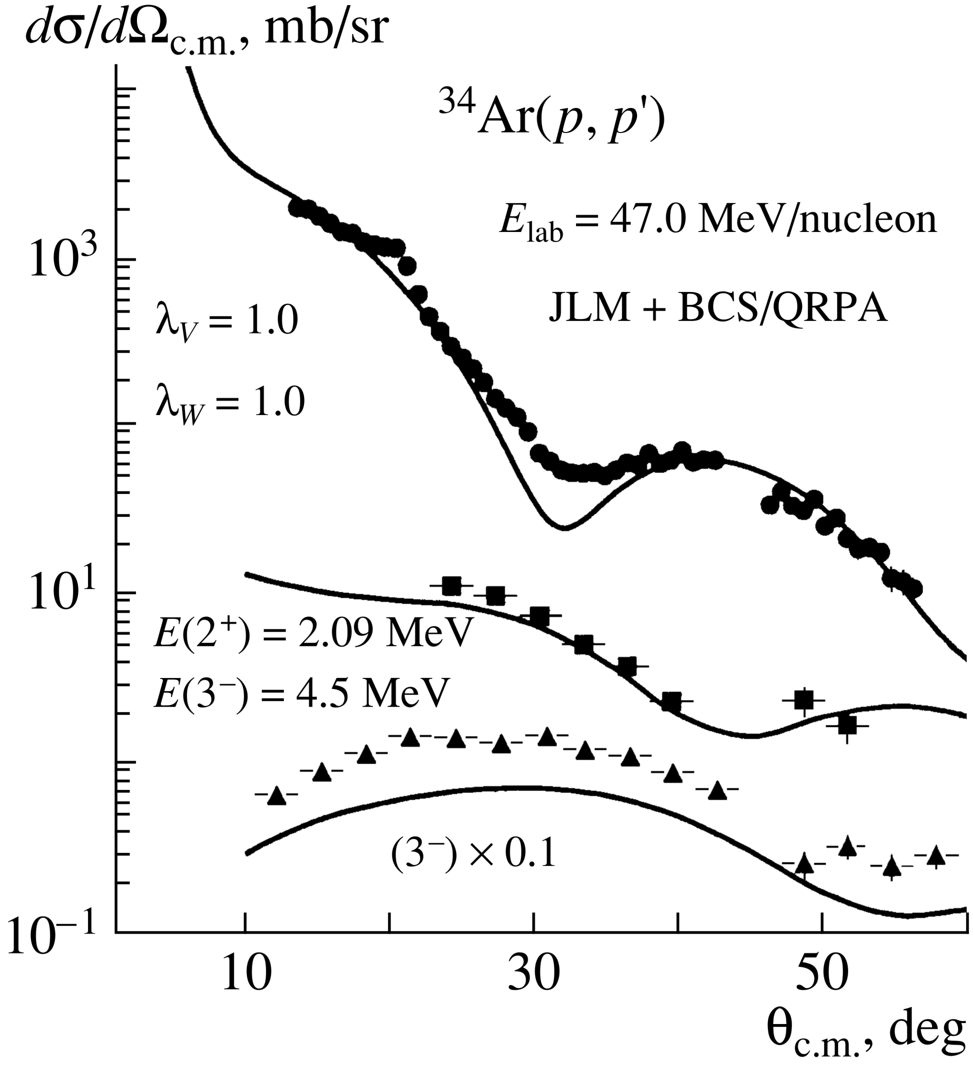
We first consider JLM as applied to proton elastic and inelastic scattering:



DWBA calculations: ^{32}S empirical densities from (e,e') + assume $\rho_n = \rho_p$. ^{38}S calculated densities.

From Lapoux, Alamanos and Khan, Phys. Atomic Nucl. **66**, 1501 (2003)

Similar calculations for p + ^{34}Ar with theoretical densities:



Agreement is also reasonable except for the 3^- state – a good test of the structure model.

JLM has the advantage that it calculates a complex OMP, so the imaginary excitation can be calculated in exactly the same way as the real.

In heavy ion work, if we use the double folding model to calculate the transition potentials as well as the diagonal OMP we are still left with the problem of what to do about the imaginary coupling potentials, since the M3Y and similar effective interactions are purely real.

One could use the DF potential for both, but there is no real physical justification for this and it places a heavy constraint on our OMP.

We therefore usually adopt a compromise: use the double folding method to calculate the real potentials (both scattering and transition) with either calculated or empirical densities (and possibly the Tassie model for the transition densities) and stick to the standard prescription for the imaginary part.

This complicates things a little if we wish to keep the real and imaginary deformations identical (either β or δ).

Alternatively we may ignore imaginary couplings and hope to include all important couplings explicitly.

The Tassie model is used to calculate the transition potential using a transition *density* derived from the ground state nuclear matter density in the following way:

$$g(r) = -\delta_L \frac{\mathrm{d}\rho(r)}{\mathrm{d}r}$$

This then replaces the ground state density in the folding model integral to produce the transition potential that we require. Note that in some codes an additional factor of $1/\sqrt{(4\pi)}$ needs to be inserted to give the correct normalisation.

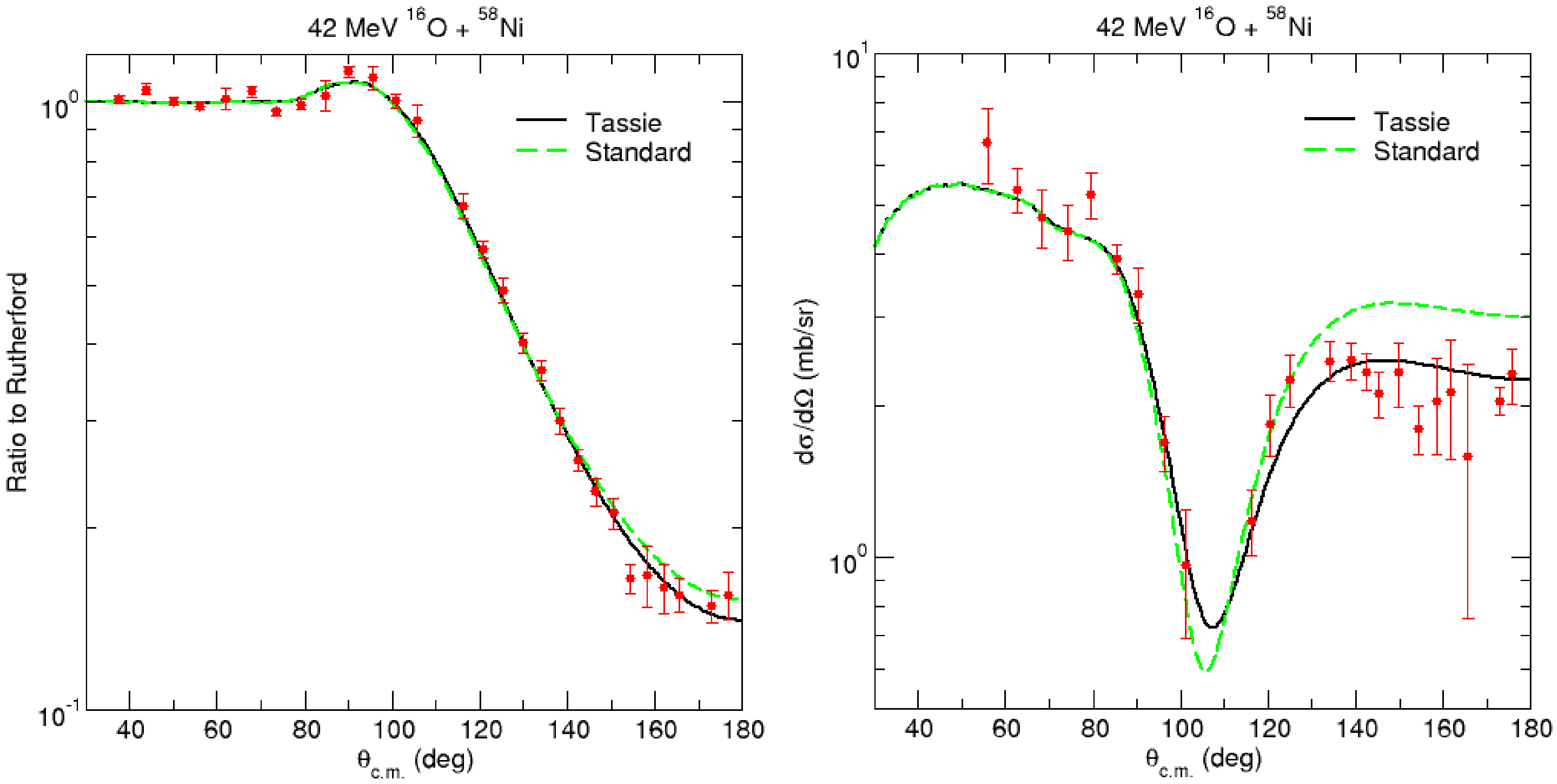
How does the Tassie model compare to the “standard” deformed potential model in an actual application?

Let us return to our example of 42 MeV ^{58}Ni(^{16}O,^{16}O')
We calculate the real transition potential in two ways:

1) Using the Tassie model by deforming the ^{58}Ni ground state density

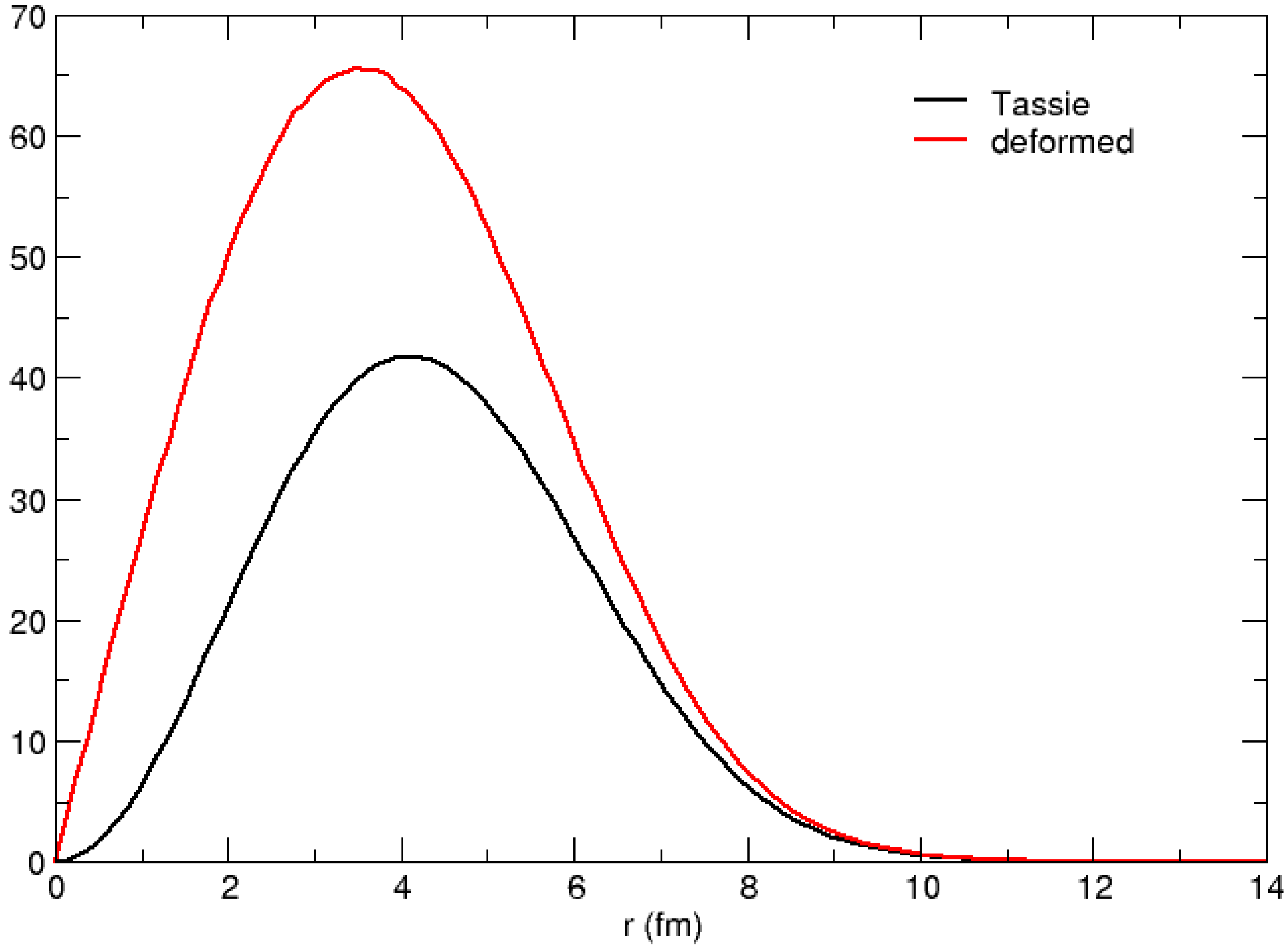
2) In the “standard” fashion by deforming the double folded real potential

In each case we use the same δ value and imaginary coupling potential (diagonal potentials are also the same in both cases).



There is clearly some minor difference, only really evident at backward angles in the inelastic scattering (N.B. the best fit δ_2 = 1.25 fm for the Tassie model)

How do the two transition potentials compare?



Rather close in the surface, where the data are sensitive to the potentials.

We see that using the double folding model to calculate transition potentials does make some difference – the δ values are usually somewhat larger than with the usual deformed potential model.

However, if the Tassie model is used to obtain the transition densities it is only first order in δ, so might not be as accurate as we could wish for strong couplings.

Again, there are choices to be made and the values we extract for deformation parameters are model dependent.

To summarise, why do we study inelastic scattering?

Inelastic scattering of the type we have been discussing here preferentially excites the strong collective excitations of nuclei, thus making it a good probe of such excitations.

The “standard” approach using a deformed potential model is problematic if we wish to draw detailed structure conclusions – not only are the deformation parameters obtained model dependent but the relation of the potential deformations to those of the density distribution is not obvious.

Nevertheless, inelastic scattering, particularly of protons, is a valuable tool for probing the nuclear response of nuclei.

For proton scattering, JLM or similar models may be used in conjunction with theoretical densities (both diagonal and transition) to calculate the inelastic scattering. In this way we can have a more direct comparison between structure theory and direct reaction data.

Such analyses can provide evidence for neutron halo effects and the breakdown of old and the appearance of new magic numbers.

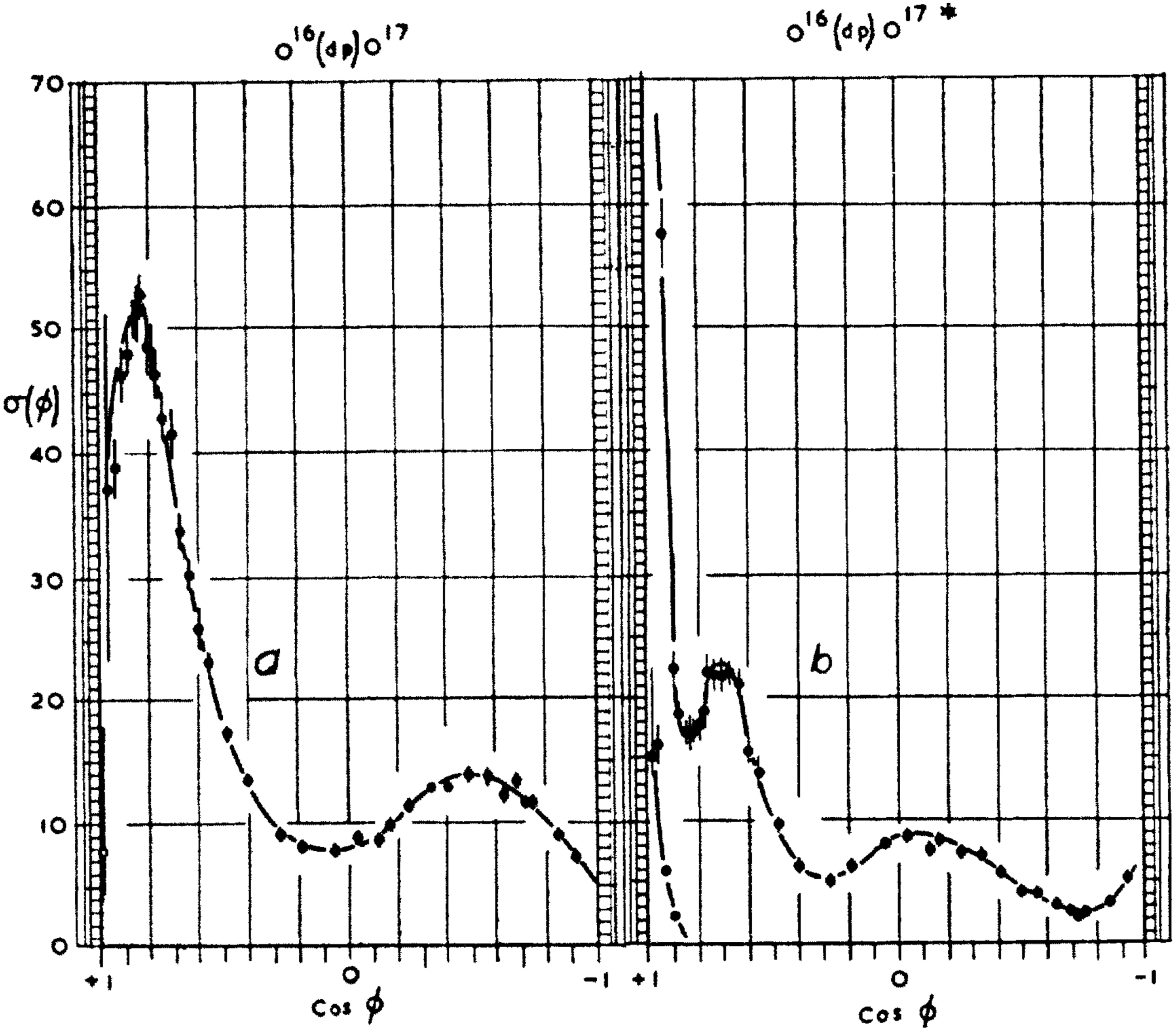
Lecture 9: transfer reactions

Transfer, or rearrangement, reactions are perhaps the most important type of direct reaction in nuclear physics. Beyond their intrinsic interest they are a valuable tool for the extraction of nuclear structure information.

Their value for structure studies arises mostly due to their *selectivity*: reactions such as (d,p) and (p,d) preferentially populate single particle and single hole states respectively.

In the right energy regime direct reactions also have a very useful characteristic …

Early on in the study of transfer reactions induced by light ions – initially deuterons – it was found that the angular distributions had characteristic shapes:



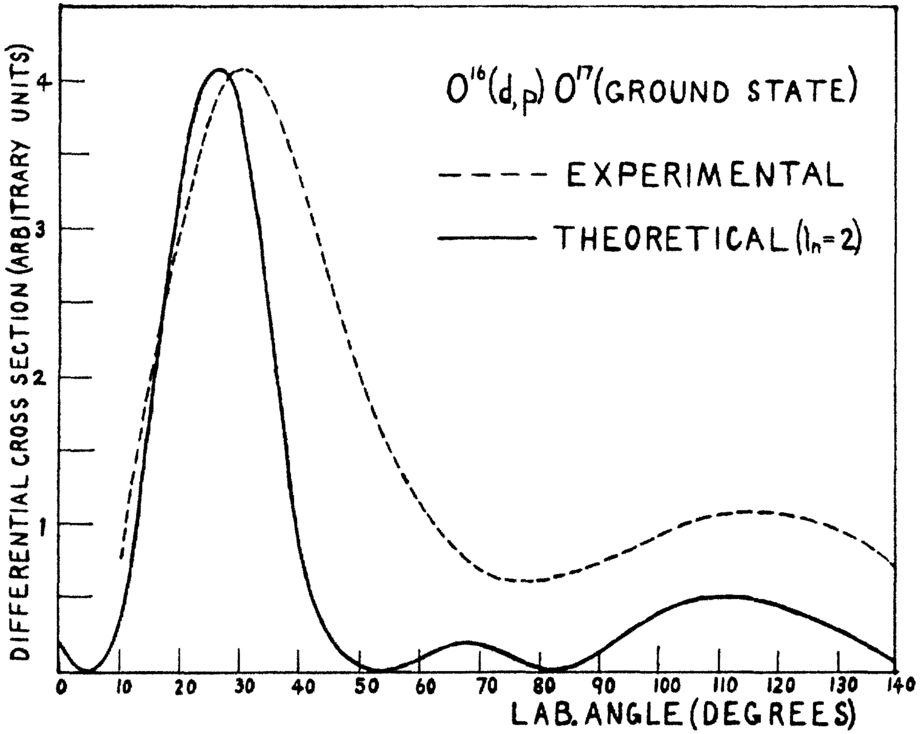
Note the forward peaked shapes common to both, but also that the shape of the AD for ^{16}O(d,p) to the g.s (a) is completely different to that to the 1st excited state (b) of ^{17}O!

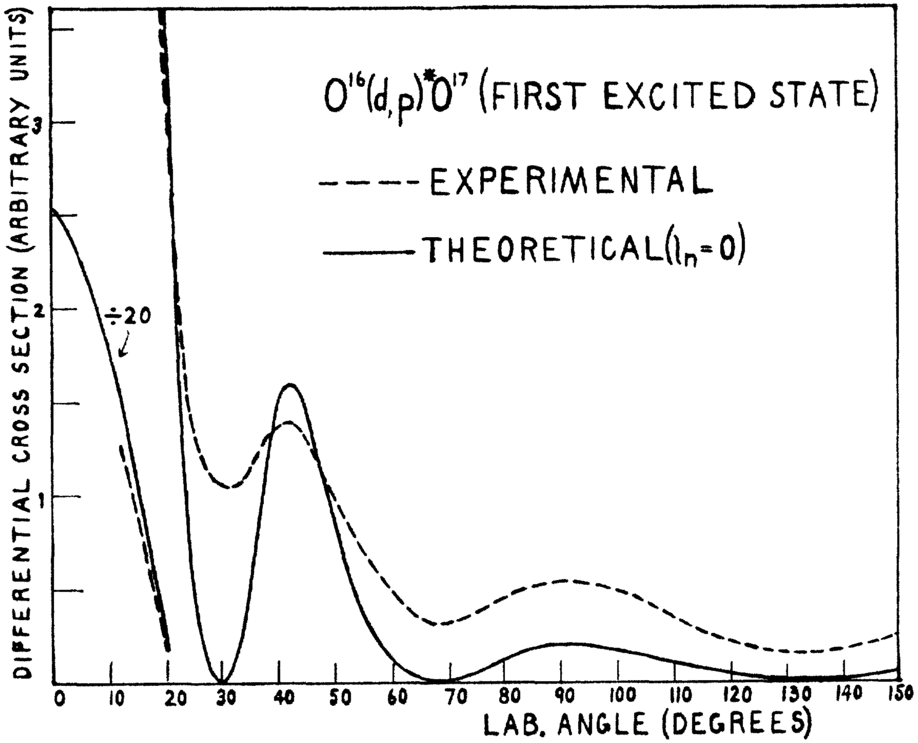
8 MeV deuteron beam from the University of Liverpool cyclotron. Gas target and nuclear emulsion detection.

Taken from Burrows, Gibson and Rotblat, Phys. Rev. **80**, 1095 (1950).

The significance was immediately realised: forward peaking = direct reaction mechanism.

Further, S. T. Butler showed that the shape of the angular distribution was directly related to the angular momentum of the transferred neutron, ℓ_n





Taken from Butler, Phys. Rev. **80**, 1095 (1950)

This had enormous consequences for the subsequent development of direct reaction studies.

As Burrows *et al*. put it in their original note: “… it appears that (d, n) and (d, p) angular distributions may be of use in determining the spins and parities of ground and excited states in many nuclei.”

It was quickly realised that Butler's theory was in fact equivalent to the plane wave Born approximation. This naturally prompted interest in examining the use of the distorted wave Born approximation (DWBA), first formulated in this context by Tobocman in 1958.

A significant problem with Butler theory was that not only were the shapes of the angular distributions only reproduced in general outline, but the magnitudes of the calculated cross sections bore no relation to the measured ones.

DWBA solved both problems. We may write the expression for the transfer angular distribution as :

$$d\sigma(\theta,E)/d\Omega = S_{JL}F_{JL}(\theta,E)$$

where $F_{JL}(\theta,E)$ is a factor containing all the reaction information (dependence on angle and energy) while S_{JL} contains structure information – the spectroscopic factor

We may calculate F_{JL} with an appropriate theory. The spectroscopic factor is then determined by normalising the calculated to the measured angular distribution. With Butler theory the S_{JL} obtained made little or no sense.

It is important to note that S_{JL} is not an observable, either in the quantum mechanical sense or in the sense that it can be measured. It is a derived quantity and therefore model dependent, a fact that should always be borne in mind. The phrase “experimental spectroscopic factor” is in fact nonsense in the literal sense.

In this lecture we shall examine how F_{JL} is calculated using DWBA.

This will involve us in presenting a few equations again, but only to illustrate some of the quantities we need to calculate.

To begin with, DWBA comes in two “flavours”: zero range and finite range. Each “flavour” can be formulated in one of two ways, known as *post* and *prior*. Which one we choose will depend on the type of reaction we wish to model.

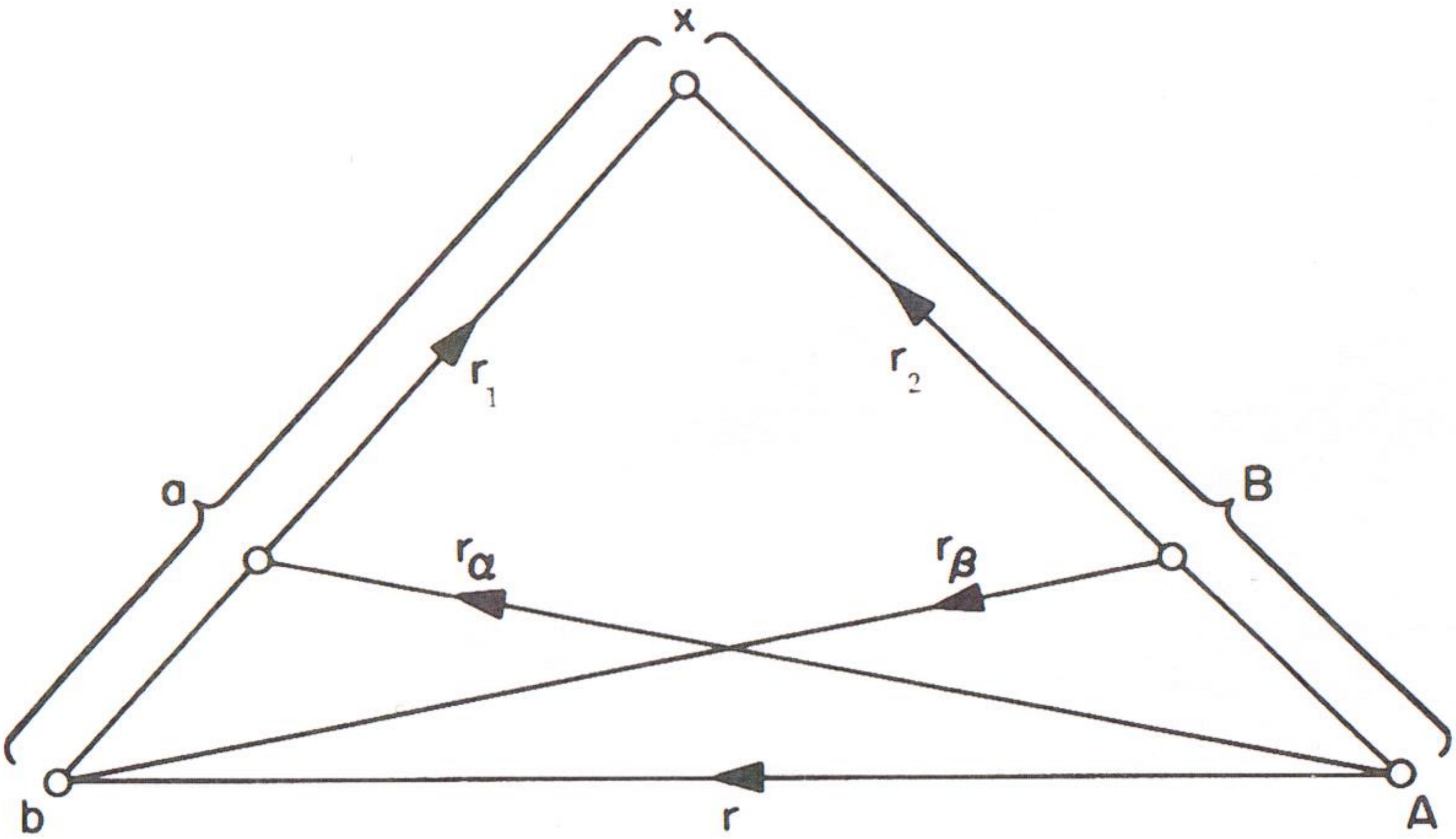
What do these expressions actually mean in terms of the formulation of DWBA?

We gave the basic expression for the DWBA transition amplitude for the reaction A(a,b)B back in lecture 3. Writing it in a slightly different notation we have:

$$\begin{aligned} T^{\mathrm{DW}}_{\beta\alpha}(\mathbf{k}_\beta,\mathbf{k}_\alpha) &= \left\langle \chi^{(-)}_\beta(\mathbf{k}_\beta)\psi_\beta \mid W \mid \chi^{(+)}_\alpha(\mathbf{k}_\alpha)\psi_\alpha \right\rangle \\ &= \iint \mathrm{d}\mathbf{r}_\beta\, \mathrm{d}\mathbf{r}_\alpha\, \chi^{(-)}_\beta(\mathbf{k}_\beta,\mathbf{r}_\beta)^*\, (\psi_\beta \mid W \mid \psi_\alpha)\, \chi^{(+)}_\alpha(\mathbf{k}_\alpha,\mathbf{r}_\alpha) \\ &= \iint \mathrm{d}\mathbf{r}_\beta\, \mathrm{d}\mathbf{r}_\alpha\, \chi^{(-)}_\beta(\mathbf{k}_\beta,\mathbf{r}_\beta)^*\, \mathrm{I}_{\beta\alpha}(\mathbf{r}_\beta,\mathbf{r}_\alpha)\, \chi^{(+)}_\alpha(\mathbf{k}_\alpha,\mathbf{r}_\alpha) \end{aligned}$$

In the second form the *nuclear matrix element* is separated from the distorted waves; the round brackets denote integration over all variables independent of $\mathbf{r}_\alpha$ and $\mathbf{r}_\beta$ and $\psi_\alpha = \psi_a\psi_A$, $\psi_\beta = \psi_b\psi_B$

The coordinate system used for the transfer reaction A(a,b)B where a = b + x and B = a + x is defined as follows:



W is the residual interaction and may be written in one of two forms. In *post form* it is:

$$W_\beta \equiv V_{bB} - U_\beta = V_{bx} + (V_{bA} - U_\beta)$$

where U_β is the OMP in the exit channel, V_{bx} is the potential that binds x onto b to form a (and which should be consistent with the wave function ψ_a) and V_{bA} is an effective interaction between b and A.

For *prior form* we have:

$$W_\alpha \equiv V_{aA} - U_\alpha = V_{xA} + (V_{bA} - U_\alpha)$$

In light ion reactions it is usual to retain only V_{bx} or V_{xA}, since the terms in brackets should cancel to a very large extent. If this is done, it is most accurate to use post for stripping and prior for pickup.

If we further put $\mathbf{r}_\beta = \lambda\, \mathbf{r}_\alpha$ we have zero range DWBA. This is equivalent to “collapsing” the coordinate system to a straight line. This is also most accurate if post and prior form are used for stripping and pickup respectively.

Zero range is adequate for light ions, indeed there are correction factors for finite range effects which, if applied, give almost identical results to full finite range DWBA.

Zero range obviates the need to evaluate a 6-dimensional integral in calculating the transition matrix, so was particularly important in the early days.

For heavy ions, only full finite range DWBA is really adequate. The neglect of the terms in brackets (sometimes called the “remnant” terms) in the expressions for W is also no longer a good approximation, although careful choice of the effective interaction V_{bA} (and the continued use of post for stripping and prior for pickup) can minimise their importance. Modern codes such as FRESCO will include them and they can be important.

Finally, we come to how a DWBA calculation is performed in practice. What are the “ingredients” that we need to assemble?

1) We need the distorted waves in the entrance and exit channels. These are calculated for us by the DWBA code from the appropriate OMPs

2) The residual interaction W. This usually boils down simply to the choice of the binding potential V_{bx} (post) or V_{xA} (prior)

3) The nuclear wave function overlaps $(\psi_{\mathrm{B}},\psi_{\mathrm{A}})$ and $(\psi_{\mathrm{a}},\psi_{\mathrm{b}})$

There is some overlap between 2) and 3) in actual applications, since the wave functions required in 3) are usually calculated by the code with binding potentials that should be consistent with the V_{bx} or V_{xA} of 2) depending on the choice of post or prior form.

Further, for 3) we also need to define 3 quantum numbers: N ℓ j. To illustrate what these mean in the most simple way we will take a specific example, the ^{12}C(d,p)^{13}C deuteron stripping reaction.

Let us suppose that we want to analyse deuteron stripping to the ground state of ^{13}C.

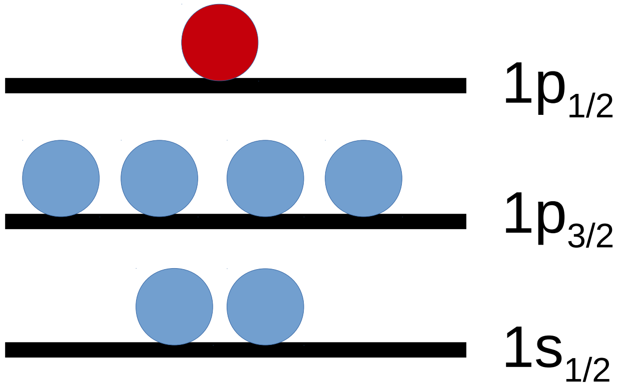
We know that the ground state of ^{13}C has spin and parity $I^{\pi} = ½^{-}$. In our standard nomenclature, the core nucleus for this state is A = ^{12}C in its 0^{+} ground state and the transferred particle x is a neutron, of intrinsic spin and parity $½^{+}$.

We therefore have $B = A + x \rightarrow {}^{13}C = {}^{12}C + n$

If we now apply the rules governing addition of angular momentum in quantum mechanics, we see that to form ^{13}C in a $½^{-}$ state the neutron must have an angular momentum relative to the ^{12}C core $\ell = 1$ (we can exclude $\ell = 0$, which could also give us a ^{13}C spin of ½, since even ℓs give + ve parity).

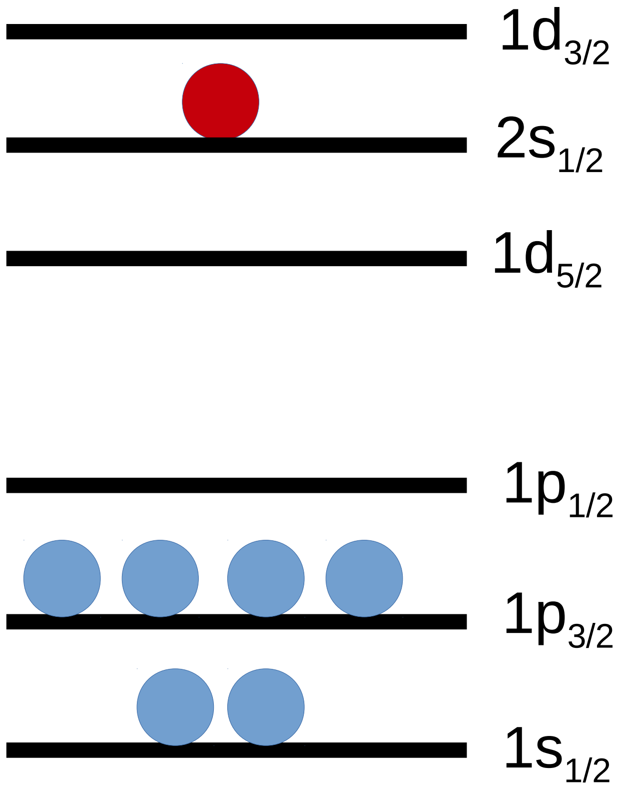
So far, so good, but what is N and how do we fix its value?

N is the number of nodes in the radial wave function for the <^{13}C|^{12}C + n> overlap. It is also the principal quantum number of the shell model level that the transferred neutron is placed in to form the desired state in ^{13}C, in this case the ½$^-$ ground state:



We see that the obvious choice is the vacant $1p_{1/2}$ orbital, so we finally have N = 1, ℓ = 1, j =½

We can perform a similar exercise for the excited states of ^{13}C. For example, the first excited state at an excitation energy of 3.089 MeV is known to have spin-parity $I^{\pi} = 1/2^{+}$, which may be formed by placing the transferred neutron in the vacant $2s_{1/2}$ orbital:



States like these are known as single particle levels and are preferentially populated in direct reactions, one of the main plus points of their use in structure studies.

Finally, to calculate the wave function for the $<{}^{13}C|{}^{12}C + n>$ overlaps we need to specify a ${}^{12}C + n$ binding potential. This is usually a (real) Woods-Saxon well with some radius and diffuseness parameters, plus a spin-orbit part.

By convention the spin-orbit term has the same radius and diffuseness as the central Woods-Saxon well and is of the Thomas type we encountered when we considered the optical model. It has a fixed depth (often around 5-6 MeV).

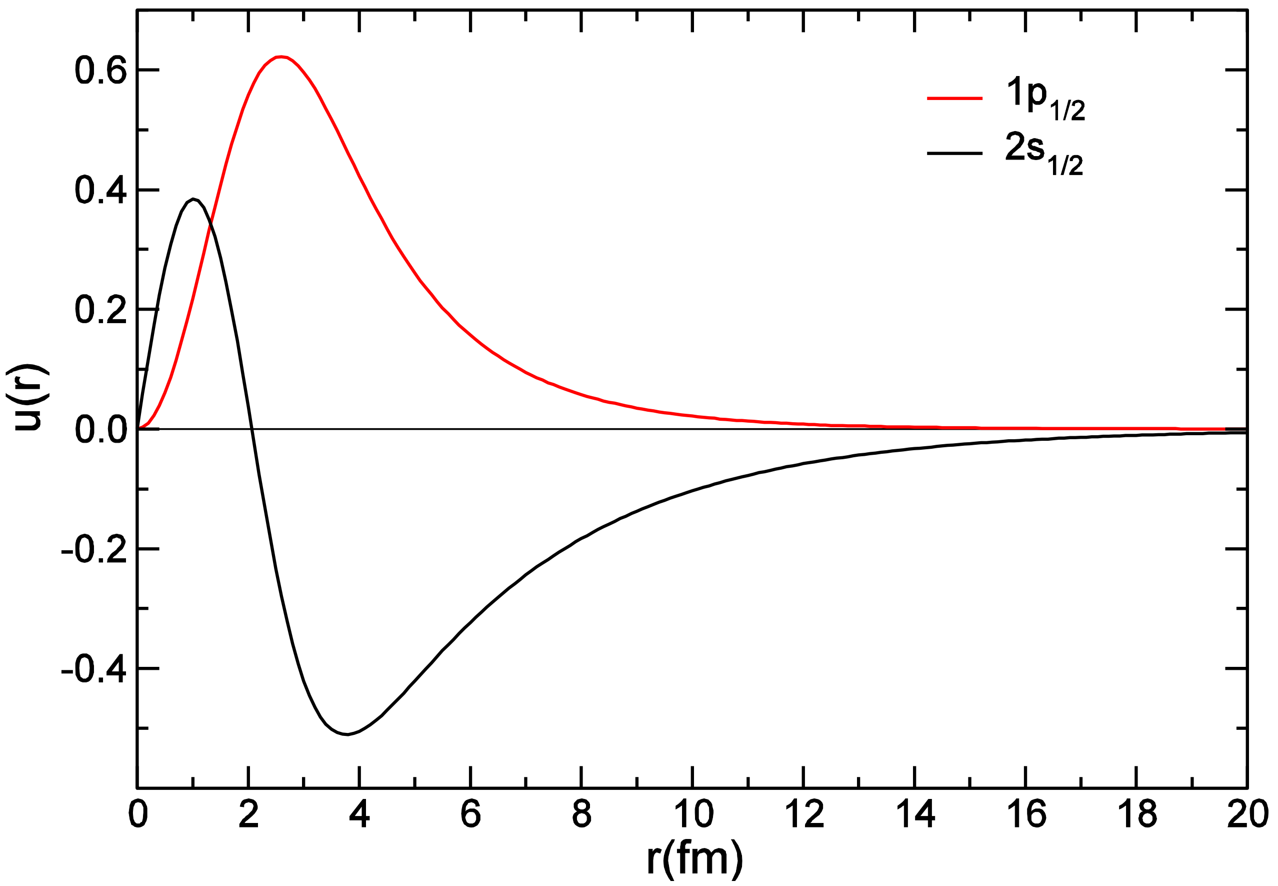
The central well-depth is adjusted so that the binding energy obtained is equal to the experimental value.

How do we choose the values for the radius and diffuseness parameters? This is an important point, as we shall see in the next lecture.

In practice, the choice is somewhat arbitrary. Common values are: $R = 1.25 \times A^{1/3}$ fm, $a = 0.65$ fm. There is no real physical justification for these time honoured values but they give reasonable results.

With the “geometry” of the binding potential fixed (and the spin-orbit term also fixed) and the $N\ \ell\ j$ values specified the reaction code will search on the well depth to give the correct binding energy.

This is what the resulting radial wave functions look like:



We must carry out the same exercise for the projectile-like overlap, in our example: <d|n + p>.

These overlaps (for light-ion induced reactions) are known, either empirically (from analyses of large sets of data) or theoretically (calculated using theoretical wave functions from structure theory).

In zero-range DWBA the projectile-like overlap is reduced to a normalisation factor D_0. Values of D_0 are well known for (d,p), (d,n), (p,d) and (n,d) reactions, less well known for (d,t) and (d,^{3}He) and very imperfectly known for reaction like (^{3}He,^{4}He).

For finite-range DWBA the full wave functions are required, as for the target-like overlaps. Again, these are well known for (d,p) etc. reactions but less well defined for others (for heavy ion projectiles we use the same procedure as for the target overlaps).

We now have all the ingredients we need to perform a DWBA calculation and we shall look at how this is done in detail in the next lecture, continuing with the $^{12}C(d,p)^{13}C$ reaction as our example.

Lecture 10: transfer reactions continued

In this lecture we shall look at how a DWBA analysis is performed in reality, taking the $^{12}C(d,p)^{13}C$ reaction at E_d = 30 MeV as our example.

Good data exist at this energy from Ohnuma *et al.*, Nucl. Phys. A **448** (1986) 205. We will look at transfer to four states in ^{13}C: the 0.0 MeV $1/2^-$, the 3.09 MeV $1/2^+$, the 3.68 MeV $3/2^-$ and the 3.85 MeV $5/2^+$.

For direct, one-step transfer these states represent a neutron in the $1p_{1/2}$, $2s_{1/2}$, $1p_{3/2}$ and $1d_{5/2}$ shell model orbitals outside the ^{12}C 0^+ core.

In this case the spins and parities of the final states are all known, so calculating the wave functions for the $<^{13}C|^{12}C + n>$ overlaps is simple. We just need to choose the following:

1) An entrance channel d + ^{12}C OMP

2) Exit channel p + ^{13}C OMPs (since the energy of the proton ejectile will be a function of E_x of the final state in ^{13}C)

3) n + ^{12}C and n + p binding potential parameters for finite range DWBA, D_0 for zero range

For 1) and 2) we simply take the same potentials as in the original analysis of Ohnuma *et al*. (often the energy dependence of the exit channel OMP is ignored and we use that for E_x = 0 MeV)

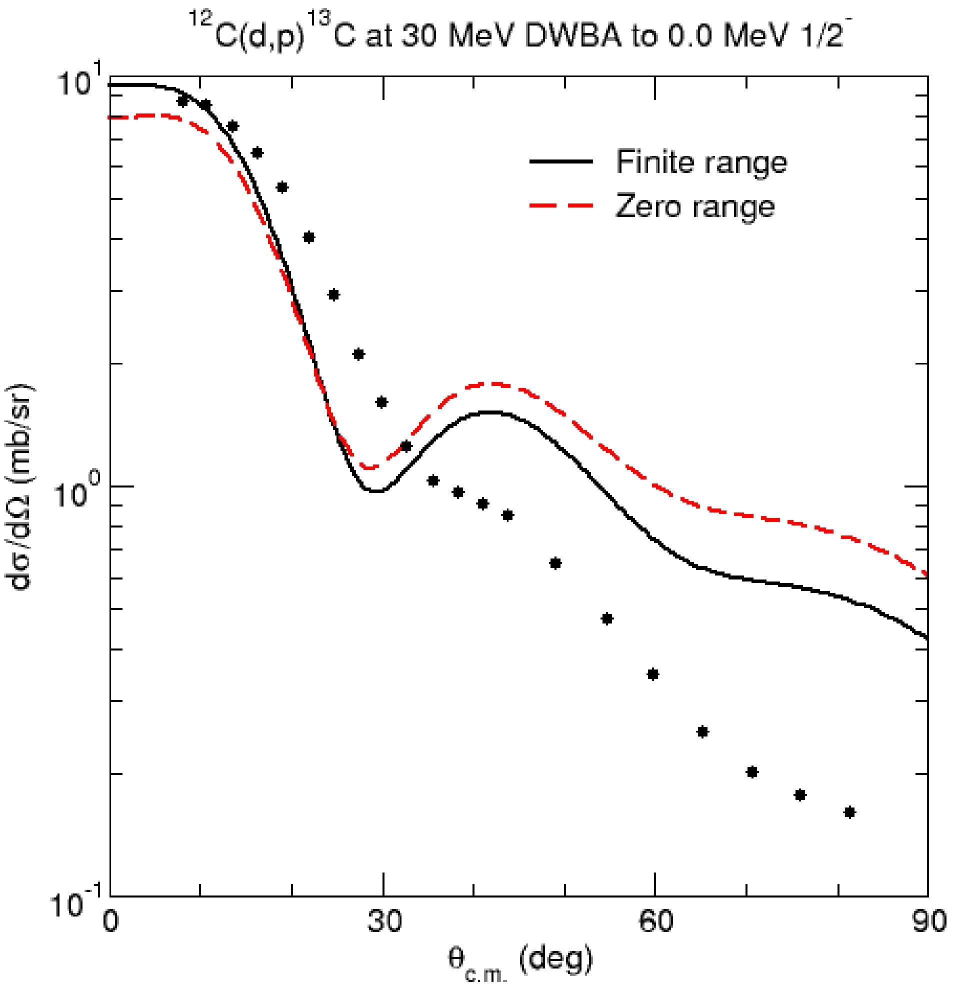
For the n + ^{12}C binding potentials, we assume a Woods-Saxon shape and radius parameters of $R = 1.25 \times 12^{1/3}$ fm, diffusenesses $a = 0.65$ fm, plus a spin-orbit term of Thomas form with a depth of 6.0 MeV.

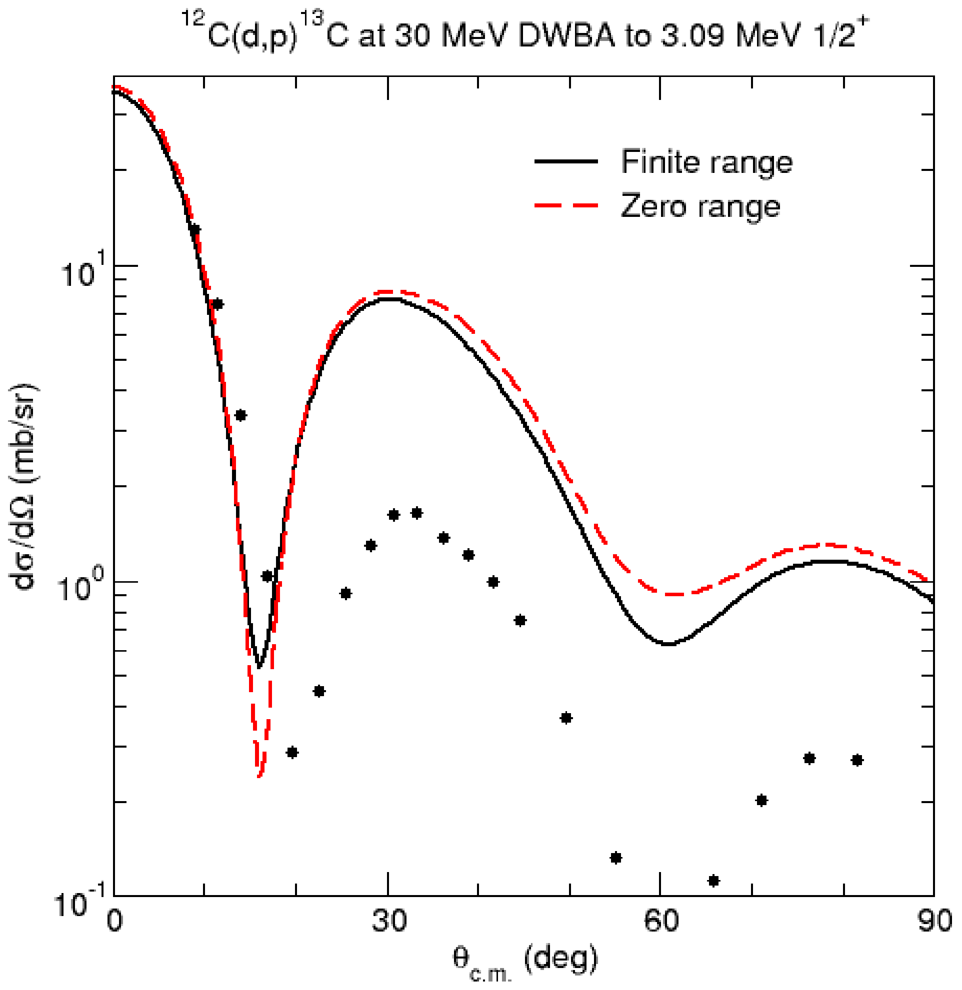
The depth of the central Woods-Saxon well is automatically adjusted by our DWBA code – FRESCO here – to give us the correct binding energies

For the n + p binding potential in finite range DWBA we use that due to Reid, Ann. Phys. (N.Y.) **50**, 441 (1968). This gives a realistic deuteron wave function, including the small D state component.

For zero range DWBA we take $D_0 = 1.55 \times 10^4$ MeV fm^3 and a finite range correction factor of 0.621 fm, a popular choice since it is given as an example in the DWUCK4 manual (a zero range DWBA code).

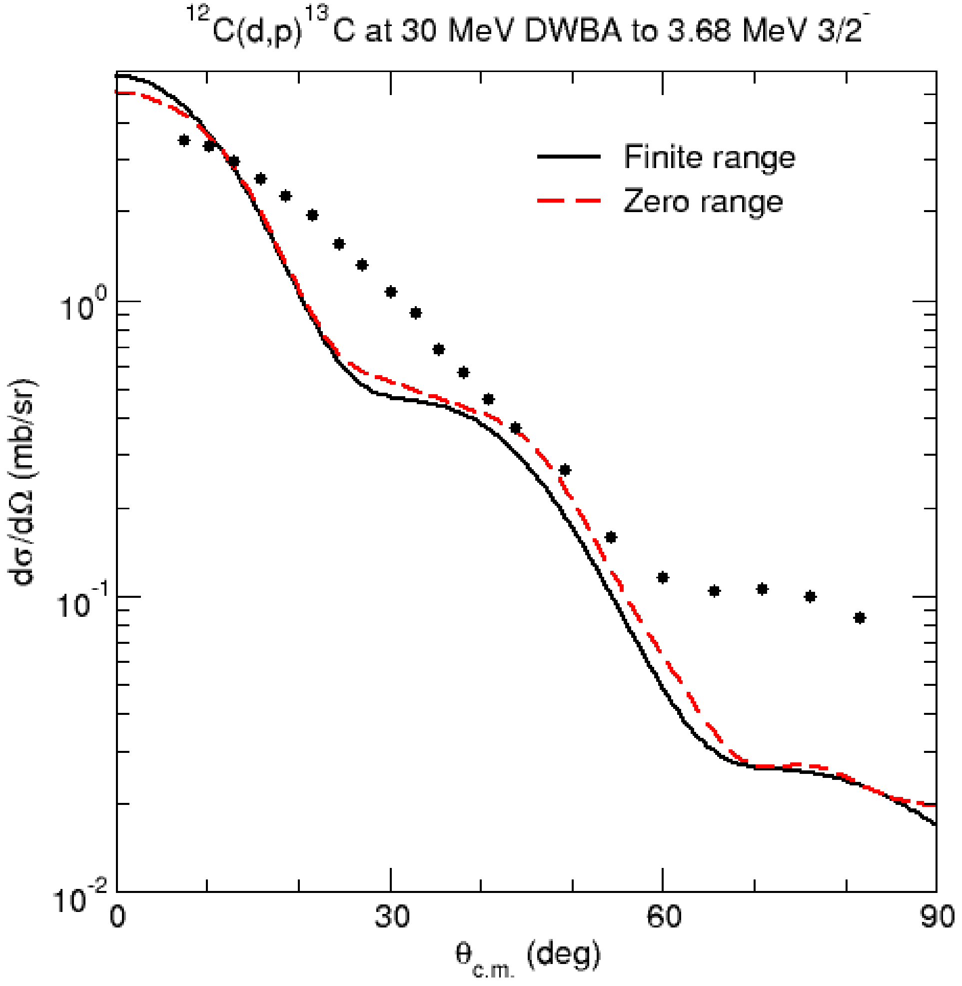
How do finite range and zero range DWBA compare with each other and the data?



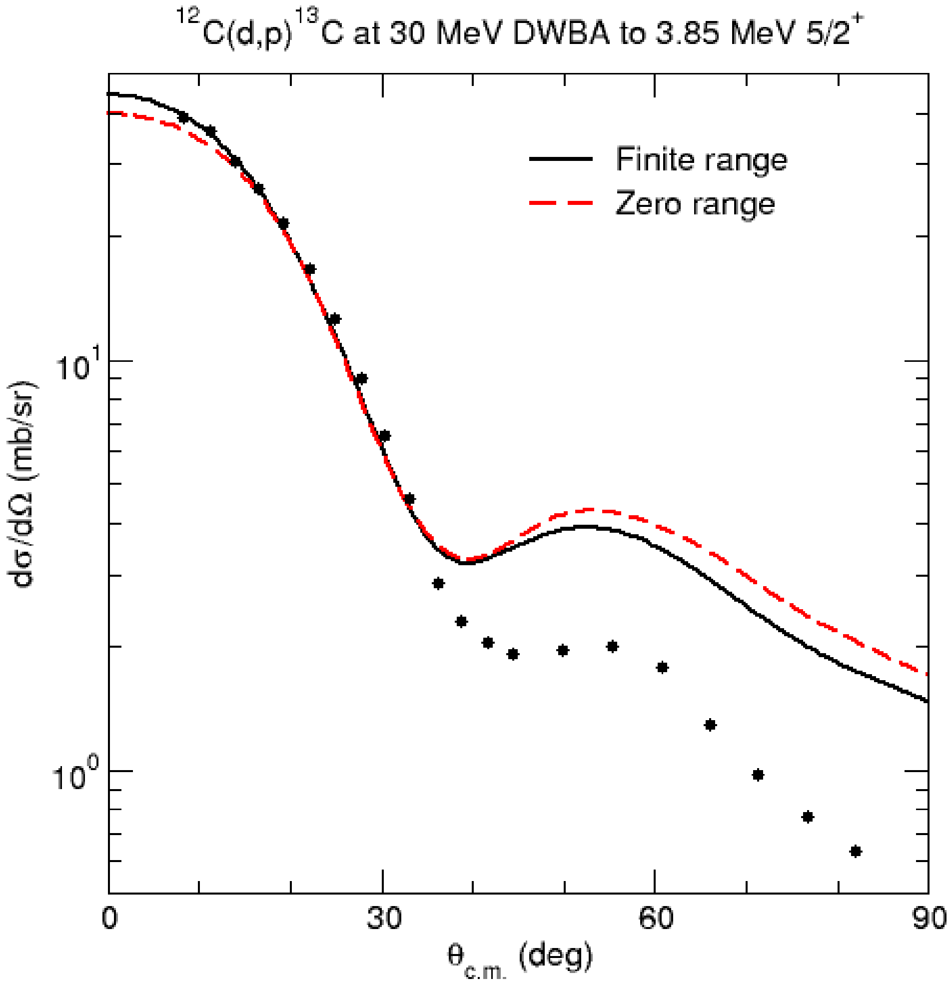


$C^2S = 0.77$

$C^2S = 1.62$



$C^2S = 0.14$



$C^2S = 0.67$

While zero range and finite range give similar results – the spectroscopic factors (C^2S) are the same for both calculations in each case – it has to be said that they do not reproduce the measured ADs very well. This could be for multiple reasons:

1) Our choice of OMP in entrance and exit channel

2) Our choice of binding potential parameters

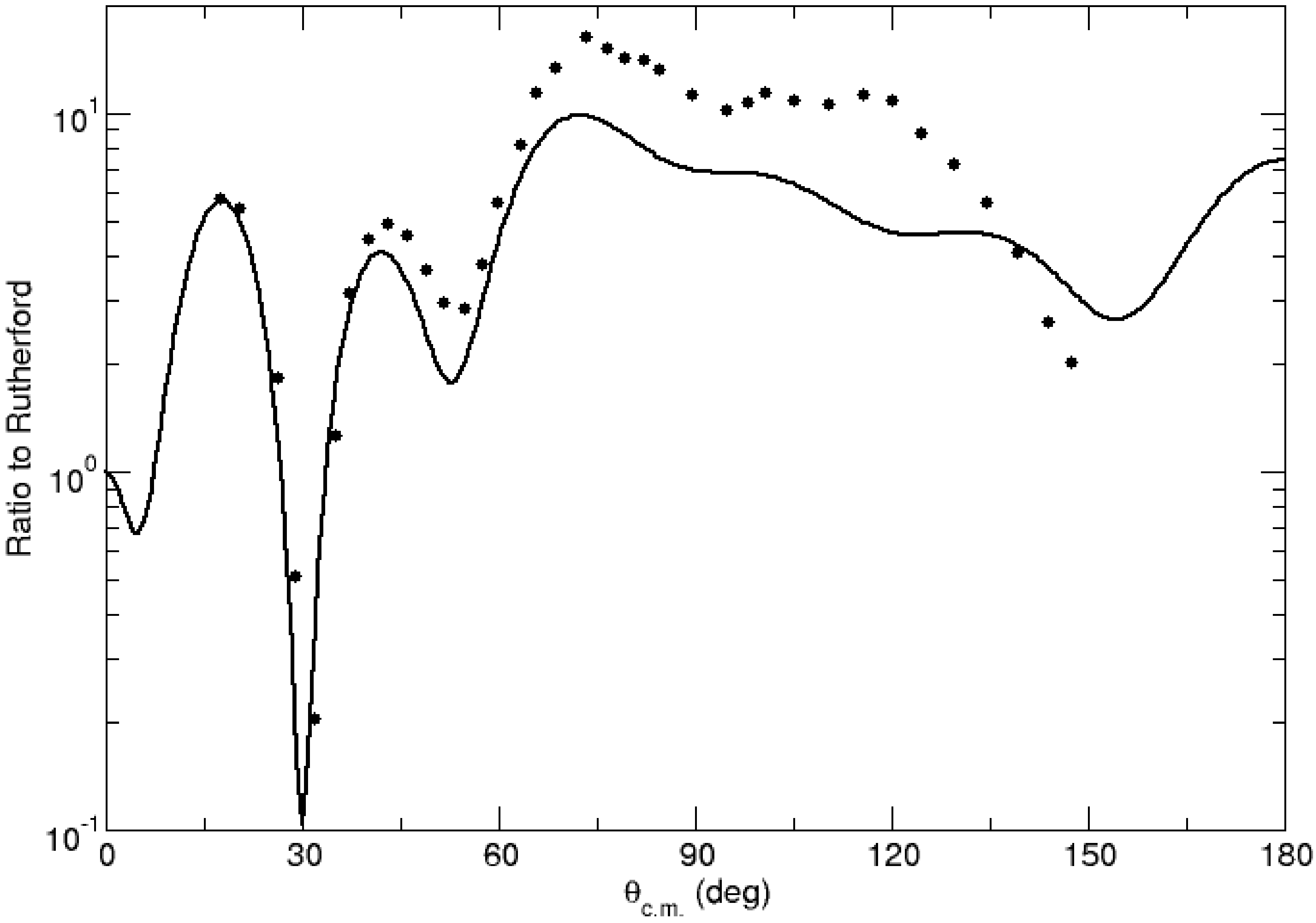
3) DWBA is not an adequate theory here, the reaction mechanism is more complicated than simple one-step transfer

Leaving aside 3) for the present (although the C^2S values suggest that it may be relevant, since none of them are particularly close to 1.0, the expected value for a pure single particle state) let us consider 1).

There are no d + ^{12}C elastic scattering data at 30 MeV but there are some at E_d = 29.5 MeV – close enough! How well are they described by our OMP?

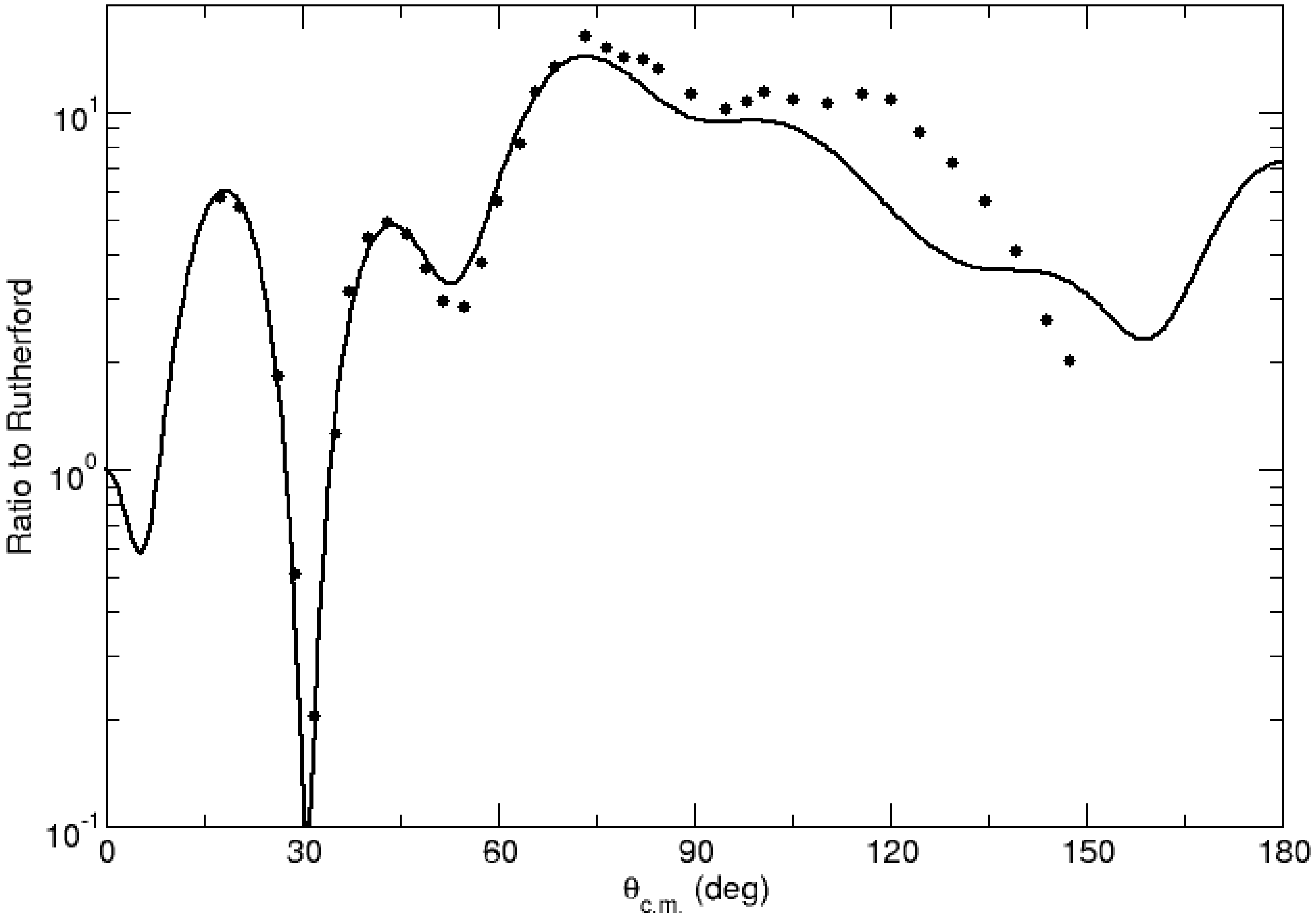
The answer is not too bad, but could do much better!

We should be able to improve on the description of the d + ^{12}C elastic scattering, but will this affect the ^{12}C(d,p)^{13}C results?

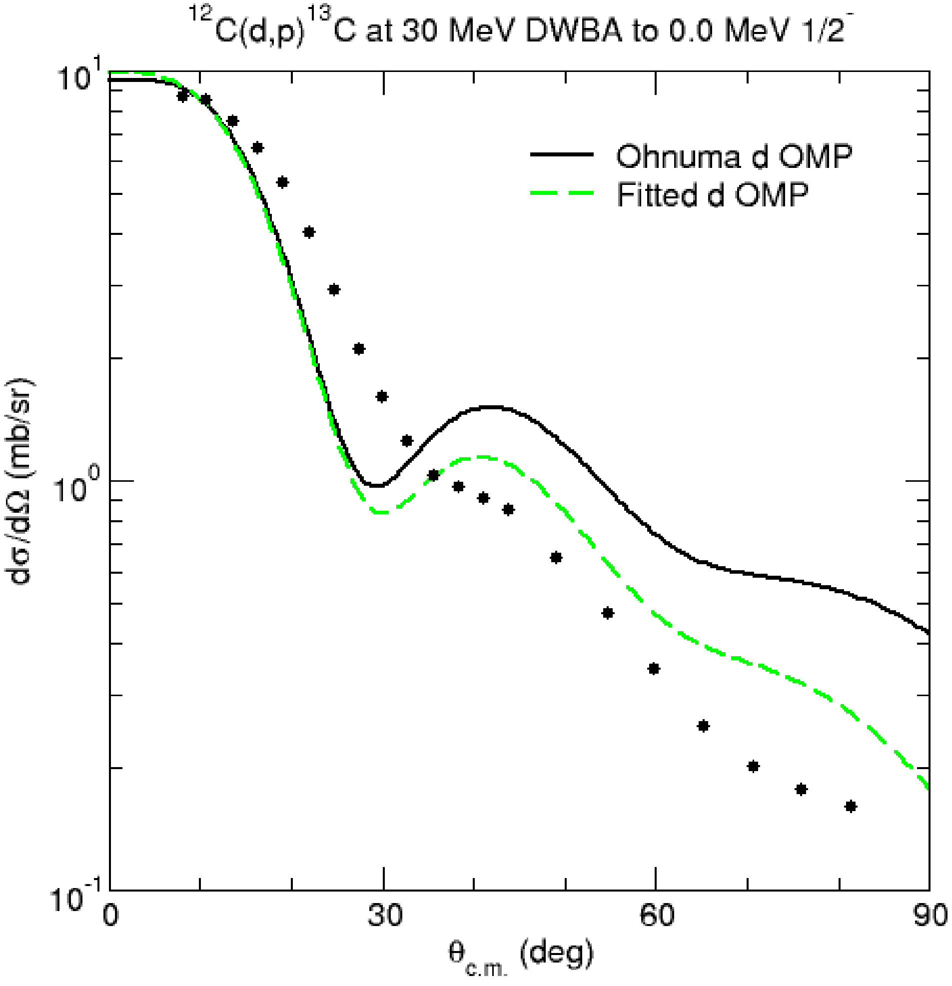


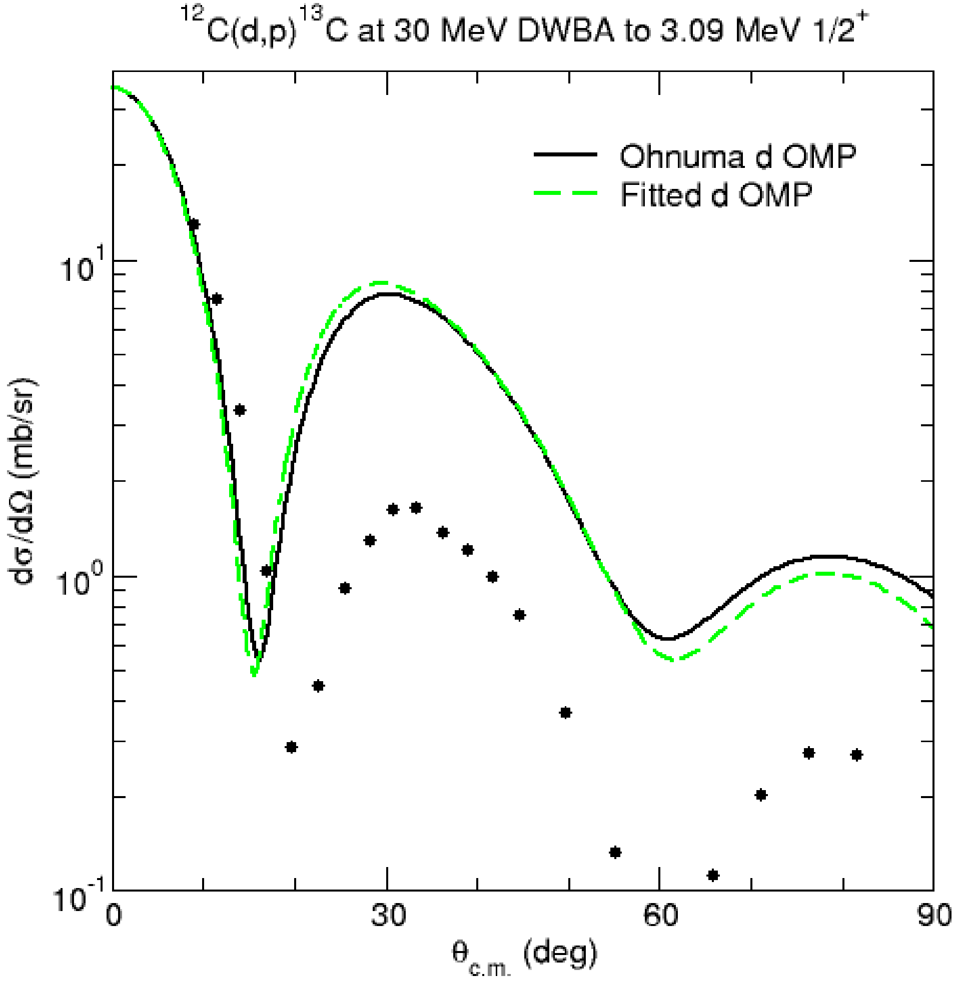
Measured 29.5 MeV d + ^{12}C elastic scattering plus result of Ohnuma *et al*.'s OMP

Improved fit to E_d = 29.5 MeV d + ^{12}C elastic scattering

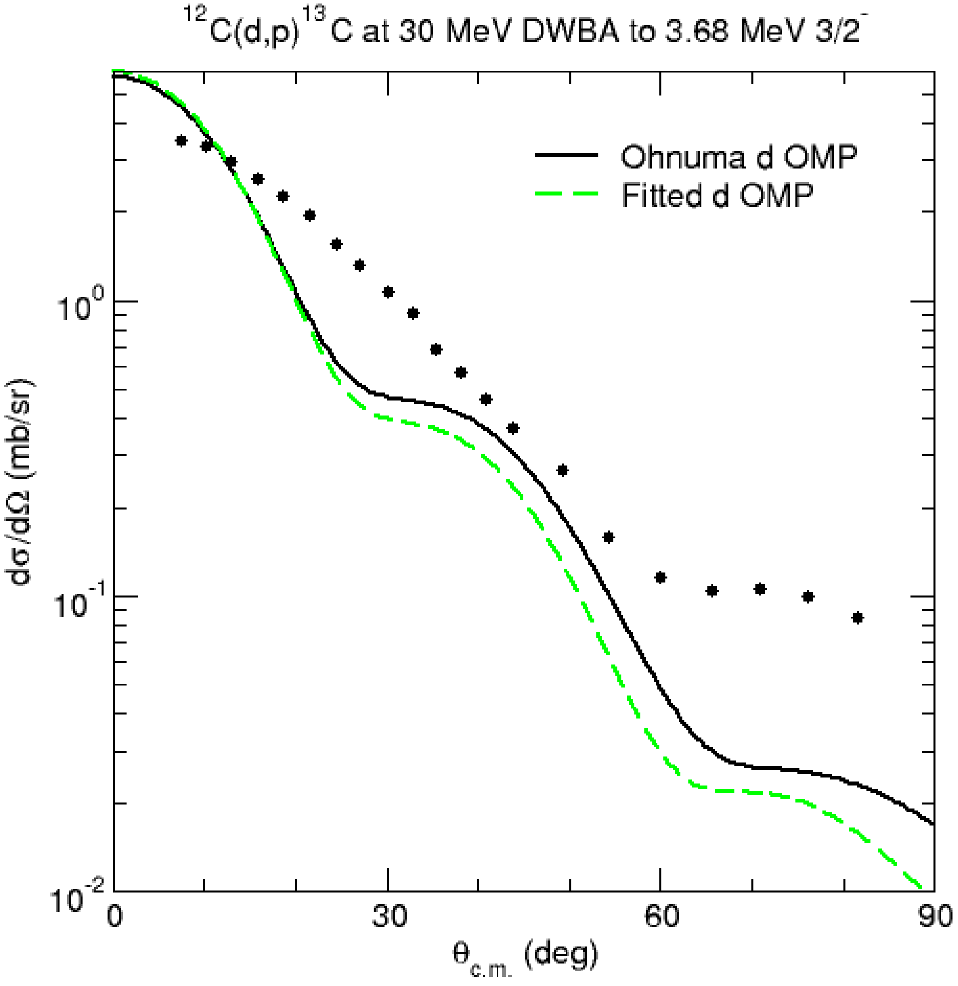


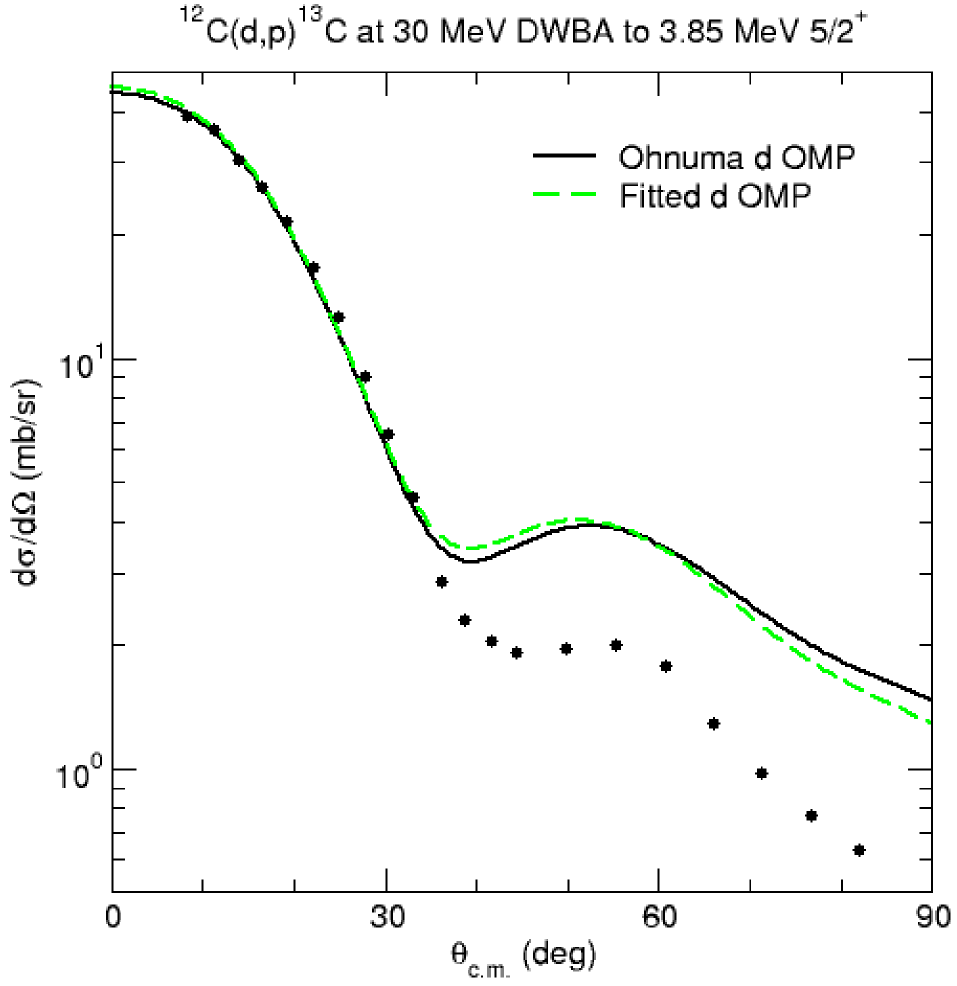
What impact does it have on the DWBA calculations?





Some slight improvement in shape for $1/2^-$ state and C^2S values change, but not significantly better


$^{12}C(d,p)^{13}C$ at 30 MeV DWBA to 3.68 MeV $3/2^-$

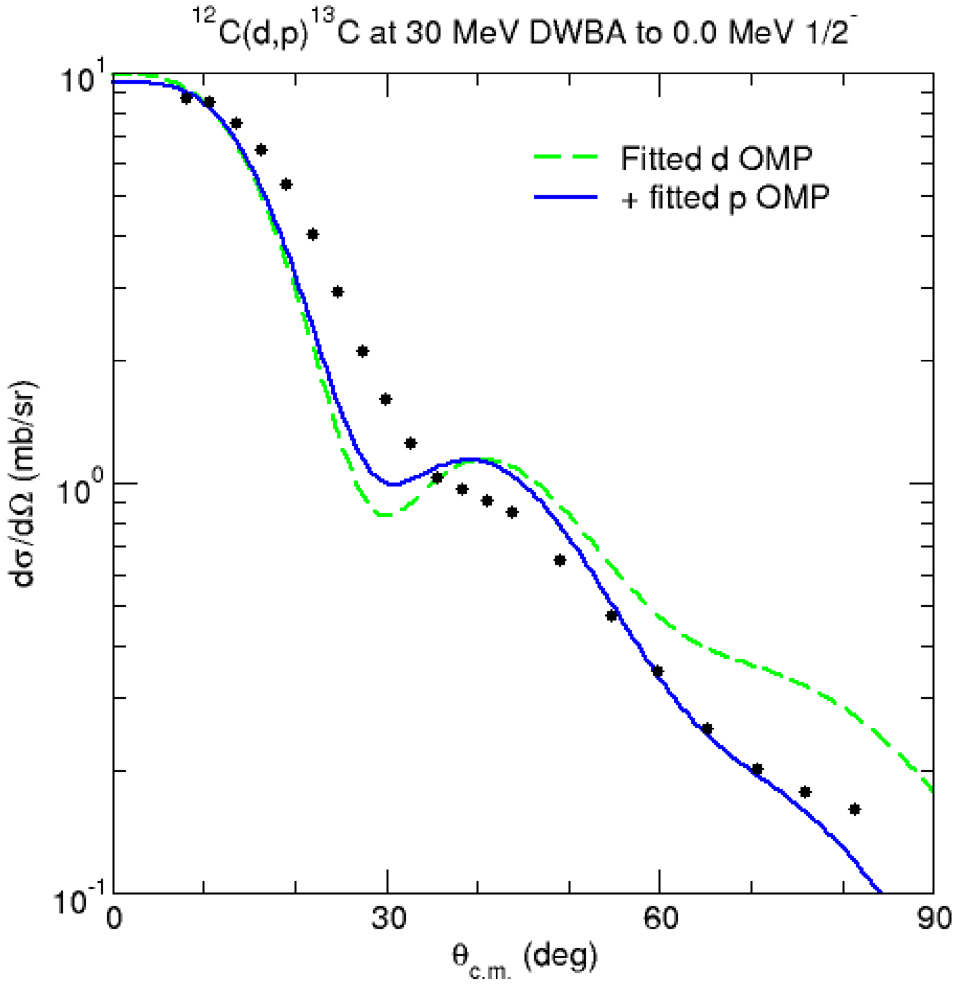

$^{12}C(d,p)^{13}C$ at 30 MeV DWBA to 3.85 MeV $5/2^+$

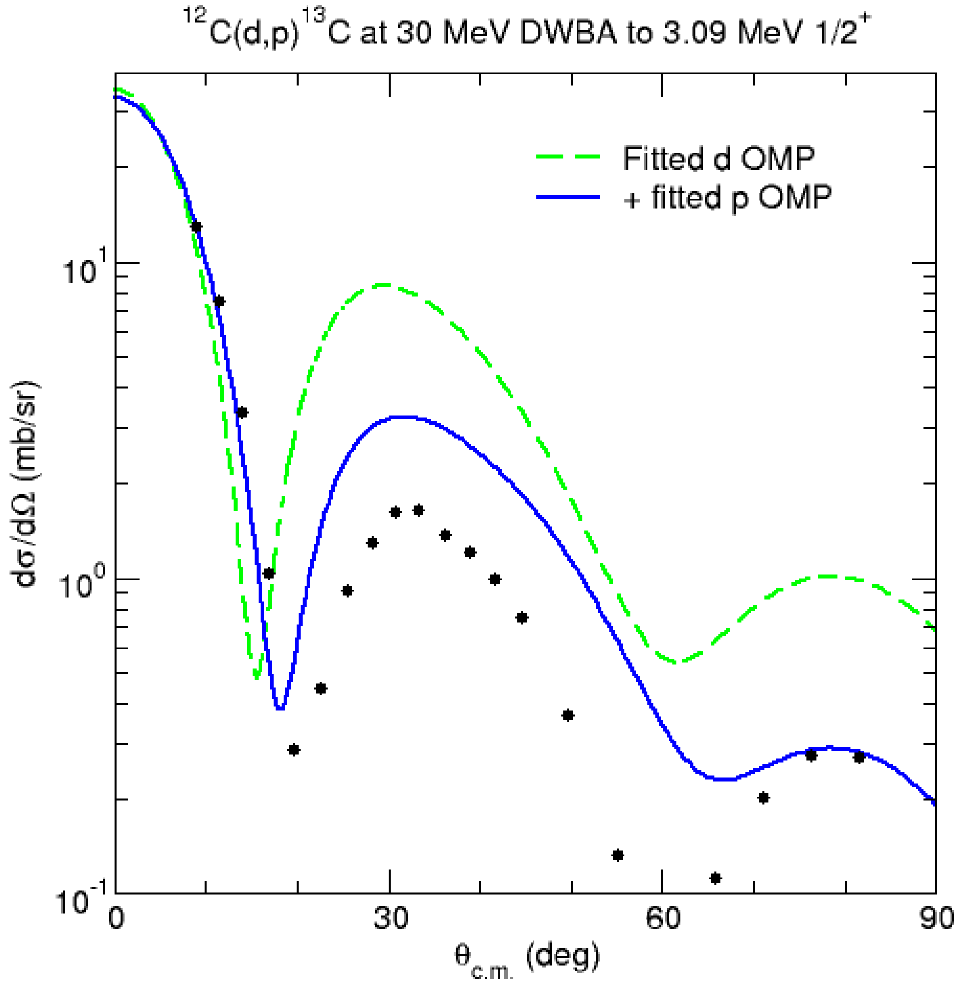
No change to C^2S values but shapes no better (slightly worse for $3/2^-$ state)

Thus, the general conclusion is that the DWBA description of the $^{12}C(d,p)^{13}C$ data is not improved by a better description of the entrance channel elastic scattering

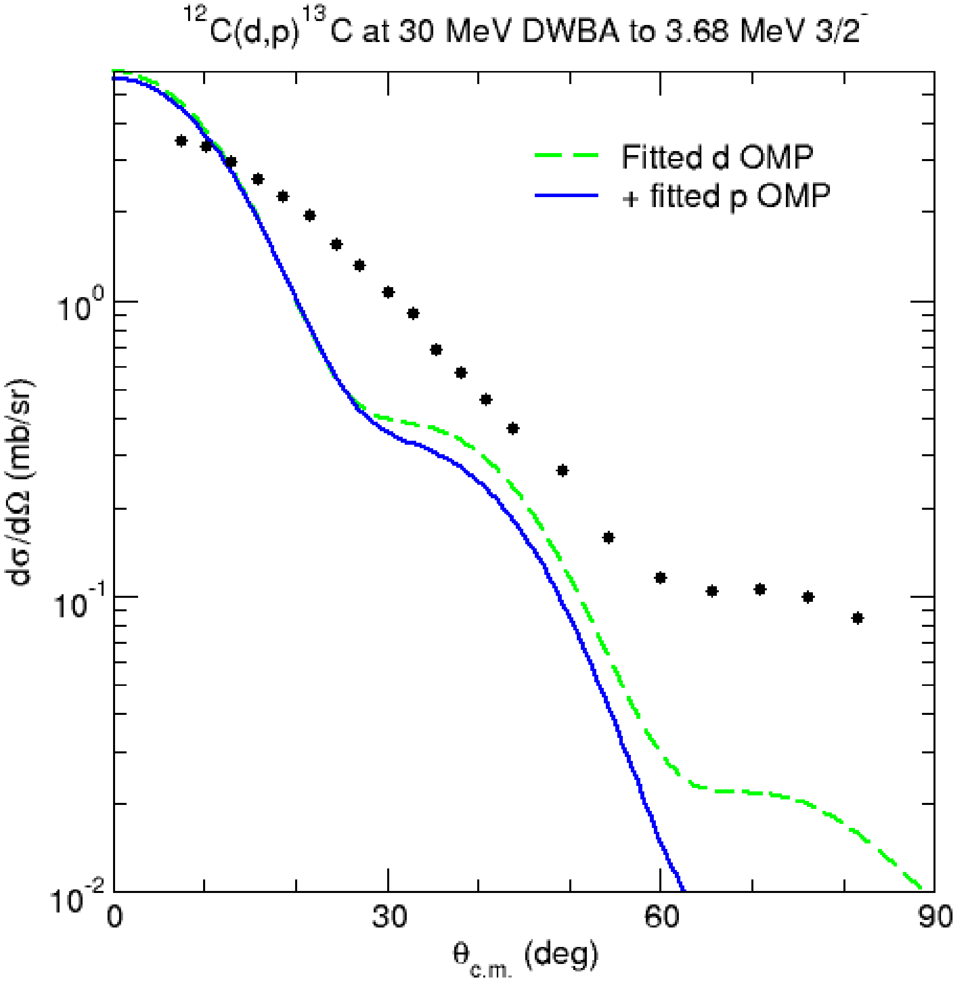
What about the exit channel? There are data for elastic scattering of 30.4 MeV protons from ^{13}C, close to the energy of the exit channel for populating the ^{13}C $1/2^-$ ground state

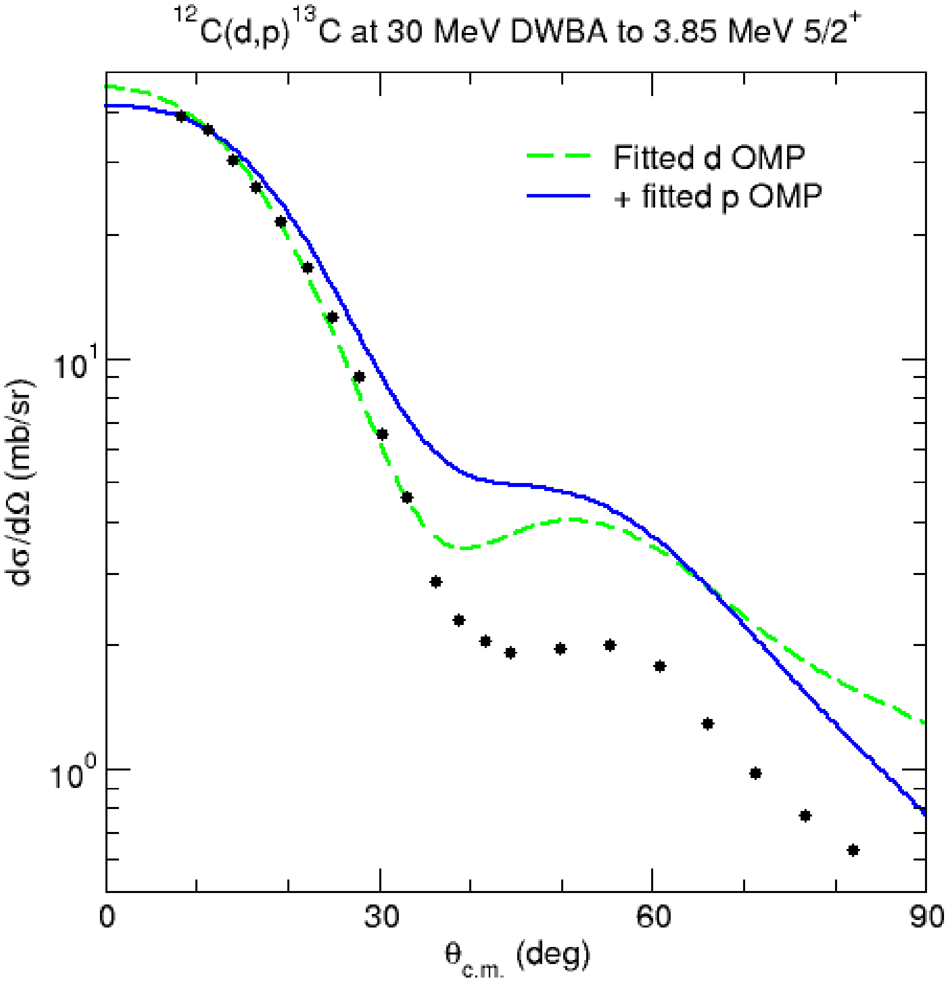
What effect does replacing the exit channel OMP of Ohnuma *et al.* have on the DWBA? We retain the “best fit” d + ^{12}C OMP in the entrance channel …





Considerable improvement in shape of ADs, but still far from a good description



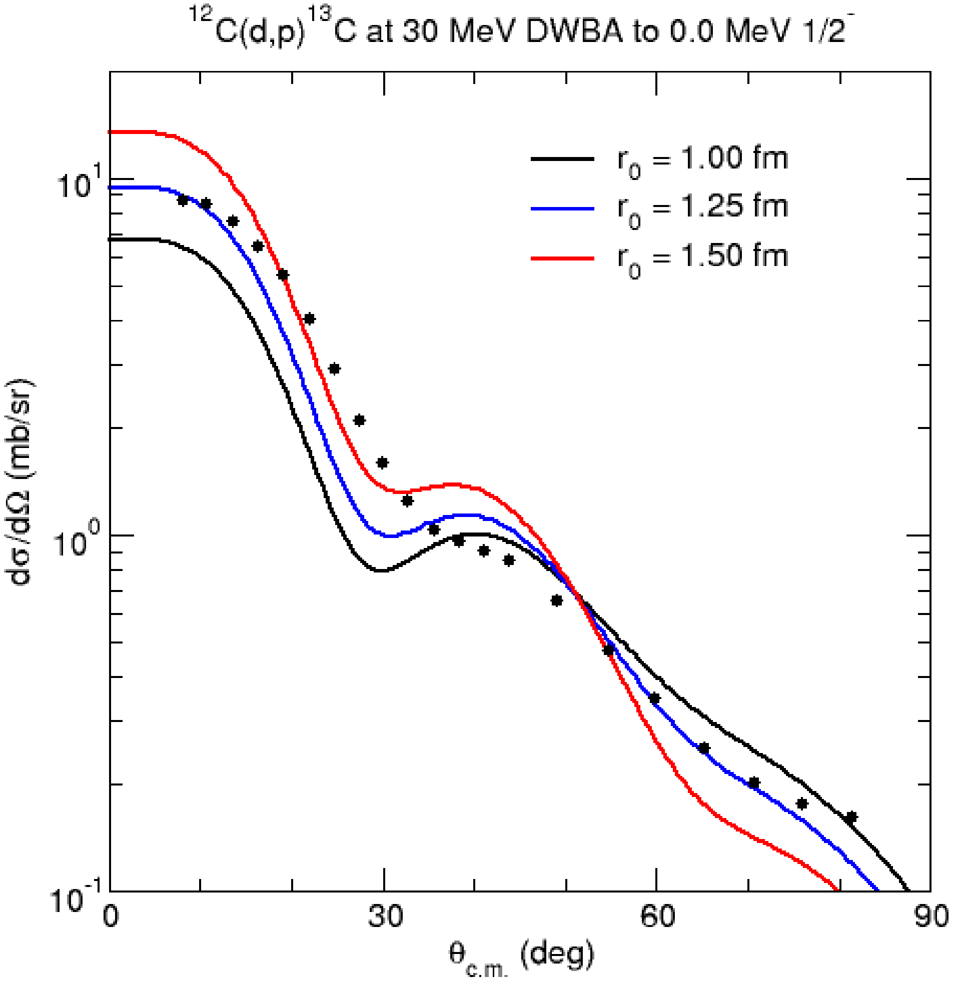


Shapes of ADs slightly worse, if anything

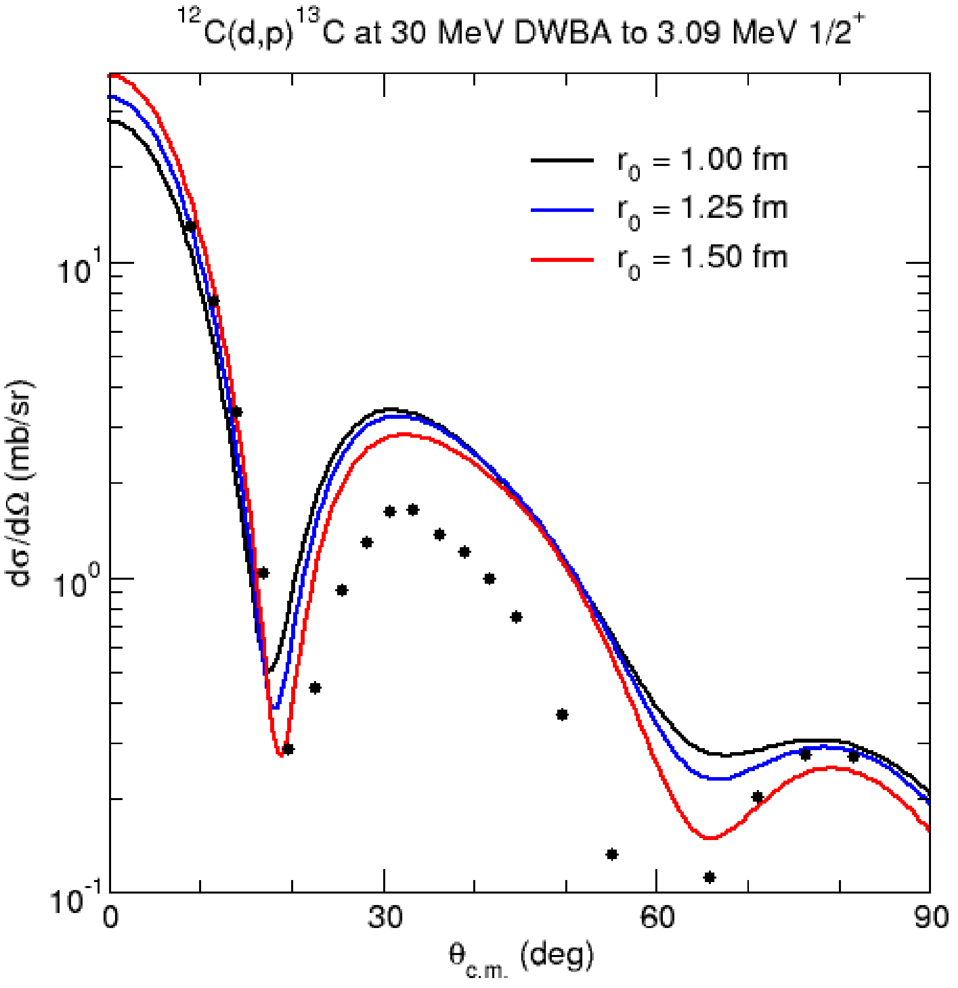
A general conclusion for this case seems to be that we can obtain some improvement by using OMPs that give a better description of the relevant elastic scattering (here mostly the exit channel) but we are still some way from a satisfactory description.

Let us now consider point 2), the binding potential parameters. We consider just the target-like case, since the projectile overlap is well-known (and in the zero range case is reduced to a normalisation factor anyway, D_0).

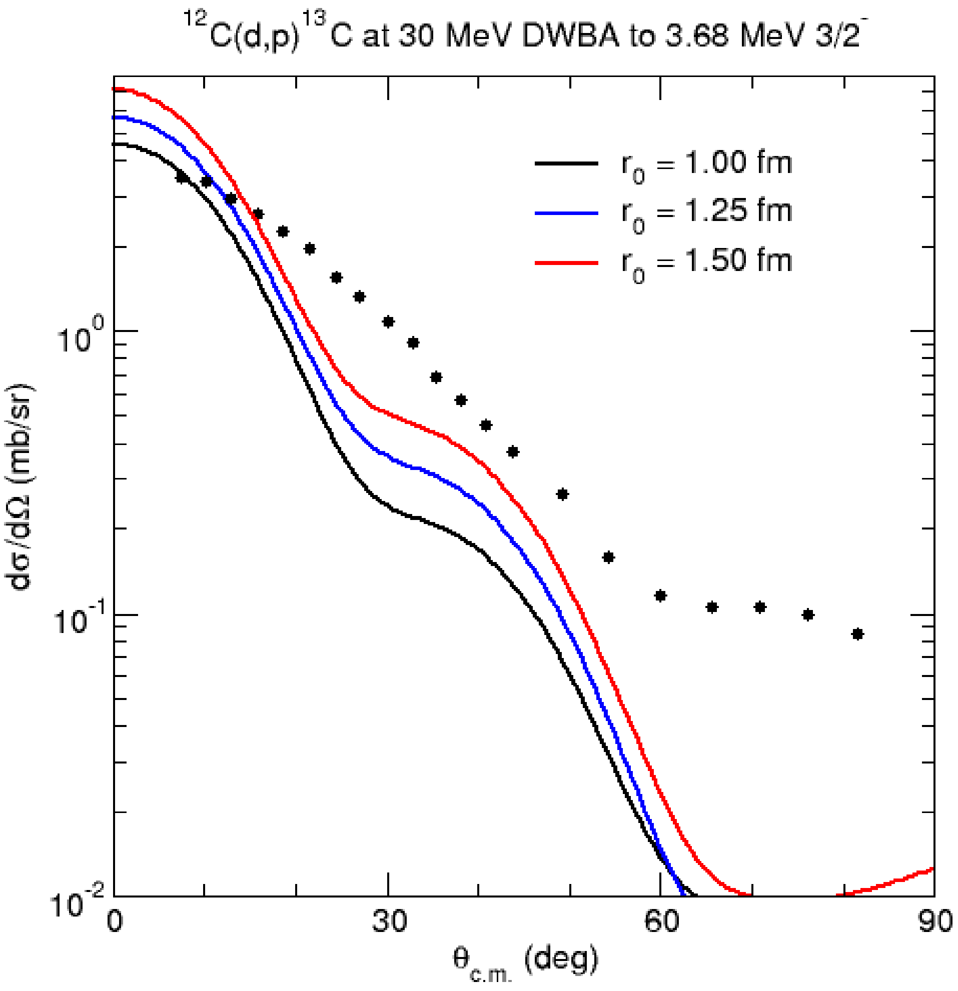
What effect does changing r_0 over a reasonable range have? Investigate effect of r_0 = 1.0 and 1.5 fm



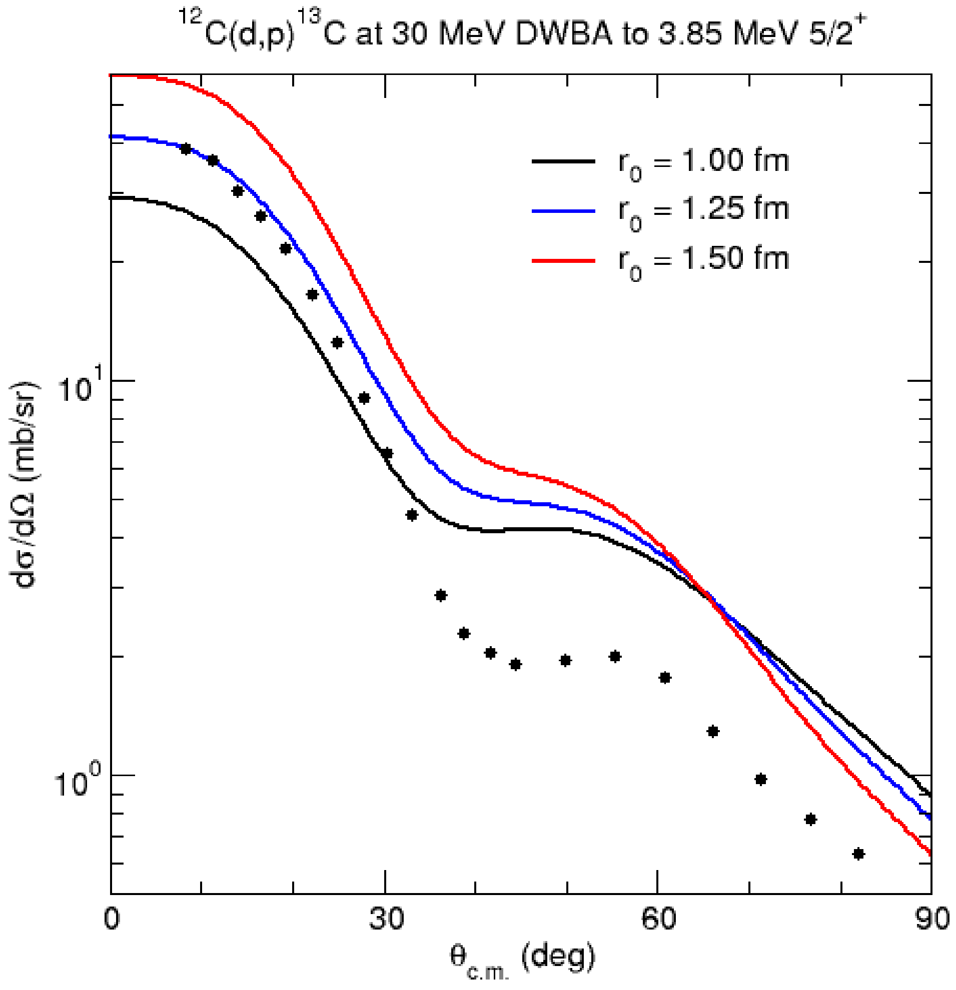
$^{12}C(d,p)^{13}C$ at 30 MeV DWBA to 0.0 MeV $1/2^-$



$^{12}C(d,p)^{13}C$ at 30 MeV DWBA to 3.09 MeV $1/2^+$



$^{12}C(d,p)^{13}C$ at 30 MeV DWBA to 3.68 MeV $3/2^-$



$^{12}C(d,p)^{13}C$ at 30 MeV DWBA to 3.85 MeV $5/2^+$

Only in the case of the $5/2^+$ state is the shape of the AD improved. The main effect in each case is to change the magnitude of the cross section, i.e. the spectroscopic factor.

This is in fact a general problem, since one of the main pieces of information that we wish to extract from direct reaction data of this type is the C^2S value. Since it is rather sensitive to the choice of r_0 and we do not have a reliable way of fixing this parameter the uncertainties can be large …

We have looked at the effect of changing the details of the input to the DWBA and it does not really solve our problem. This suggests that we need a more sophisticated theory to fit these data. Further support for this comes from the spectroscopic factors.

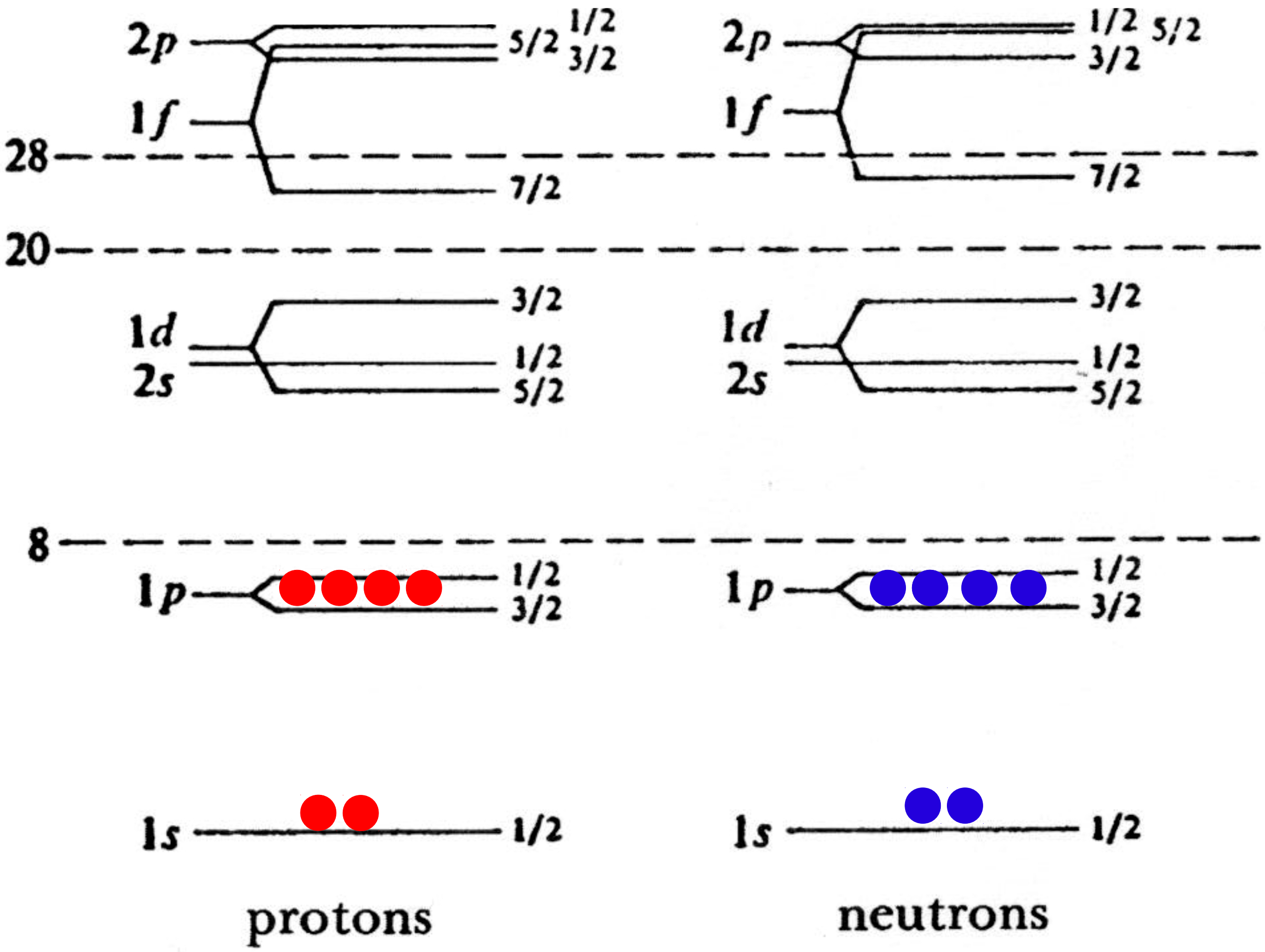
How do our extracted values compare to a shell model calculation? We compare to Cohen & Kurath, Nucl. Phys. A 101, 1 (1967) – an old but still useful calculation.

State	C^2S from (d,p)	C^2S from CK
0.00 MeV $1/2^-$	0.72	0.61
3.09 MeV $1/2^+$	1.20	
3.68 MeV $3/2^-$	0.13	0.19
3.85 MeV $5/2^+$	0.80	

Agreement is reasonable, given the poor description of the ADs by DWBA (CK did not calculate the C^2S values for the positive parity states since their model only allowed states up to the p-shell).

What is significant is that the $3/2^-$ state has a small C^2S, much smaller than 1.0. This is therefore not well described as a neutron in a shell model level outside the ^{12}C core in its ground state.

A glance at the shell model scheme explains this:



In the ground state of ^{12}C the 1p$_{3/2}$ sub-shell is full! The 2p$_{3/2}$ sub-shell is too far away to be a credible substitute, therefore the low-lying 3/2^{-} state in ^{13}C must have a more complicated structure => DWBA cannot describe transfer to this state well

The poor description of the 3/2$^-$ state by the DWBA is thus explained by its structure; there is little overlap between the ground state of ^{12}C and this state in ^{13}C.

Therefore, to describe this state (and the better to describe the other states) we need to use a more sophisticated theory of the reaction which will allow the inclusion of 2-step (or more) reaction paths, e.g. excitation of the 4.44 MeV 2$^+$ state in ^{12}C followed by transfer of a neutron to form states with structures like: $\langle {}^{13}\mathrm{C} | {}^{12}\mathrm{C}^* + n \rangle$

To do this we need to use the coupled channels Born approximation (CCBA) or the coupled reaction channels (CRC) formalisms. We shall look at these next week. We will also briefly consider the rôle of the weak binding energy of the deuteron and how it can be taken into account.

Finally for this lecture, a word about spectroscopic factors.

You will have noticed that I have used the label "C^2S" in the text of this lecture to denote the spectroscopic factor; why have I done so and what is this mysterious C?

C^2S is just another label for the quantity S_{LJ} that we met when first discussing transfer reactions. In DWBA it is simply the normalisation factor between the calculated and measured transfer angular distributions.

It is sometimes written in this fashion where the C denotes the isospin Clebsch-Gordan coefficient for the overlap. To make things easier, we keep to our example of $<^{13}C|^{12}C + n>$. For overlaps of this type C is defined as the Clebsch:

$$\left\langle \mathrm{T}^{12\mathrm{C}}\ \mathrm{T}_{\mathrm{z}}^{12\mathrm{C}}\ \mathrm{T}^{\mathrm{n}}\ \mathrm{T}_{\mathrm{z}}^{\mathrm{n}} \mid \mathrm{T}^{13\mathrm{C}}\ \mathrm{T}_{\mathrm{z}}^{13\mathrm{C}} \right\rangle$$

T and T_z are well known for the neutron: $T = T_z = ½$

For ^{12}C and ^{13}C, T_z is easily calculated by the following relation:

$$T_z = (N - Z)/2$$

This applies to all nuclei (N.B. the sign conventions for T_z are as used in nuclear physics).

For T, we must look it up for the state in question. For light nuclei, the following site gives the values:

http://www.tunl.duke.edu/nucldata/

For ^{12}C, $T_z = 0$ and we find that $T = 0$ too for the ground state.

For ^{13}C, $T_z = (7 - 6)/2 = ½$ and we find that $T = ½$ too for all the states we are considering

We therefore need the following Clebsch:

$$< 0\ 0\ ½\ ½\ |\ ½\ ½>$$

These are tabulated and there are also on-line calculators, e.g.

http://personal.ph.surrey.ac.uk/~phs3ps/cgjava.html

In this case, $C = 1.0$ so $C^2S = S$. It is not always so, and C is often < 1.0.

Why bother with this extra complication since the number we extract from our data is C^2S anyway?

Simply that often calculated spectroscopic factors are just S and do not contain the C^2 factor (this is the case for the Cohen and Kurath values cited earlier) so if we wish to compare our empirical values with, say, the shell model then we need to be aware of this and calculate the relevant C^2.

In the next lecture, more sophisticated reaction theories.

Lecture 11: transfer reactions continued

We saw last time how DWBA was unable to provide a satisfactory description of the ^{12}C(d,p)^{13}C reaction at E_d = 30 MeV.

This suggests a more complicated reaction mechanism than the simple, single step weak process supposed by DWBA.

We saw that at least in one case (the $3/2^-$ excited state) the shell model level sequence supports this possibility since the $1p_{3/2}$ level is already full in ^{12}C.

There are two main possible sources of complex, i.e. more than single step, reaction paths in the d + ^{12}C system:

i) ^{12}C has a strongly coupled excited state (2+) at E_x = 4.44 MeV. We can envisage excitation of this state followed by transfer as a two step process

ii) the deuteron is weakly bound ($S_n = S_p$ = 2.22 MeV) so breaks up easily. This process can interfere with the (d,p) transfer so absorbing its effect into the OMP may not be adequate

Of course, both these effects may contribute …

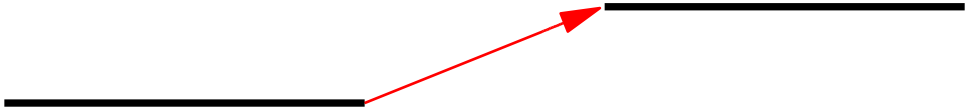
We begin by looking at the first possibility, inelastic excitation followed by transfer. How do we handle it?

The simplest way is to retain DWBA for the transfer step and to use coupled channels theory for the inelastic scattering step. This is known as the Coupled Channels Born Approximation, or CCBA

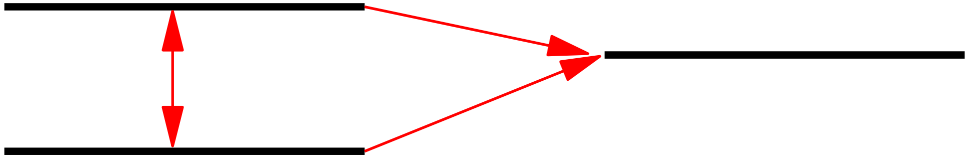
We use CC for the inelastic excitation since it is strong (therefore DWBA not in principle suitable) and retain DWBA for the transfer as it is assumed weak

How can we visualise CCBA versus DWBA?

DWBA



One transfer path. One spectroscopic factor to determine



CCBA

Two transfer paths. Two (or more) spectroscopic *amplitudes* to determine

Before considering the extra ingredients needed in CCBA compared to DWBA we should explain what we mean by a spectroscopic *amplitude* in this context and why there could be more than two (surely two transfer paths, two amplitudes?)

First of all, the spectroscopic amplitudes (SA) are the square roots of the spectroscopic factors we met last time, i.e. $SA = \sqrt{(C^2S)}$

They are therefore signed (can be +ve or −ve) and the signs matter since we may have *interference* between different reaction paths

Secondly, we may have more than two of them to determine here thanks to our old friends the rules for angular momentum addition in quantum mechanics

As an illustration, take $^{12}C(d,p)^{13}C$ to the 3.68 MeV $3/2^-$ state of ^{13}C. As we saw last time, for transfer onto the 0^+ state of the ^{12}C core the neutron can only occupy a $p_{3/2}$ shell model level. Since the core spin J_{core} was 0 it didn't enter consideration.

For the two step path, $J_{core} = 2$, so we now have to "couple" to the core spin too when we determine which shell model levels can give us a $3/2^-$ state in the final ^{13}C nucleus.

Let us see how it works: we have a ^{12}C core in its 2^+ state and we wish to form a ^{13}C in a $3/2^-$ state by transferring a neutron onto the ^{12}C core.

The neutron must have odd ℓ with respect to the ^{12}C core to give a −ve parity ^{13}C state, so it has to have $\ell = 1$ or 3 here (5 is too large).

Starting with $\ell = 1$. We can have $j = 1/2$ or $3/2$ (since $1 + 1/2 = 3/2, 1/2$ quantum mechanically)

1/2 can couple to 2 (the core spin) to give: 5/2, 3/2
3/2 can couple to 2 to give: 7/2, 5/2, 3/2, 1/2
We can have components like: $^{12}\mathrm{C}\ 2^+ \otimes 1p_{1/2}$
$^{12}\mathrm{C}\ 2^+ \otimes 1p_{3/2}$

Following the same procedure for $\ell = 3$:
we can have j = 7/2 or 5/2

5/2 can couple to 2 to give: 9/2, 7/2, 5/2, 3/2, 1/2
7/2 can couple to 2 to give: 11/2, 9/2, 7/2, 5/2, 3/2

We can have components like: $^{12}\mathrm{C}\ 2^+ \otimes 1f_{5/2}$
$^{12}\mathrm{C}\ 2^+ \otimes 1f_{7/2}$

Thus we now (in principle) have *five* amplitudes
to determine instead of one spectroscopic factor
− allowing more reaction paths comes at a price …

What else do we need? The entrance channel OMP is now replaced by a coupled channels calculation exactly as we did previously. We have a deformed optical potential that explicitly includes the $0^+ \rightarrow 2^+$ coupling in ^{12}C.

Take the B(E2) value from the literature (it is well known) and ideally fix the nuclear deformation length by fitting d + ^{12}C inelastic scattering data at E_d = 30 MeV (data exist at 28 MeV – close enough).

Exit channel OMP remains unchanged. We just need to specify the extra reaction paths in the input file.

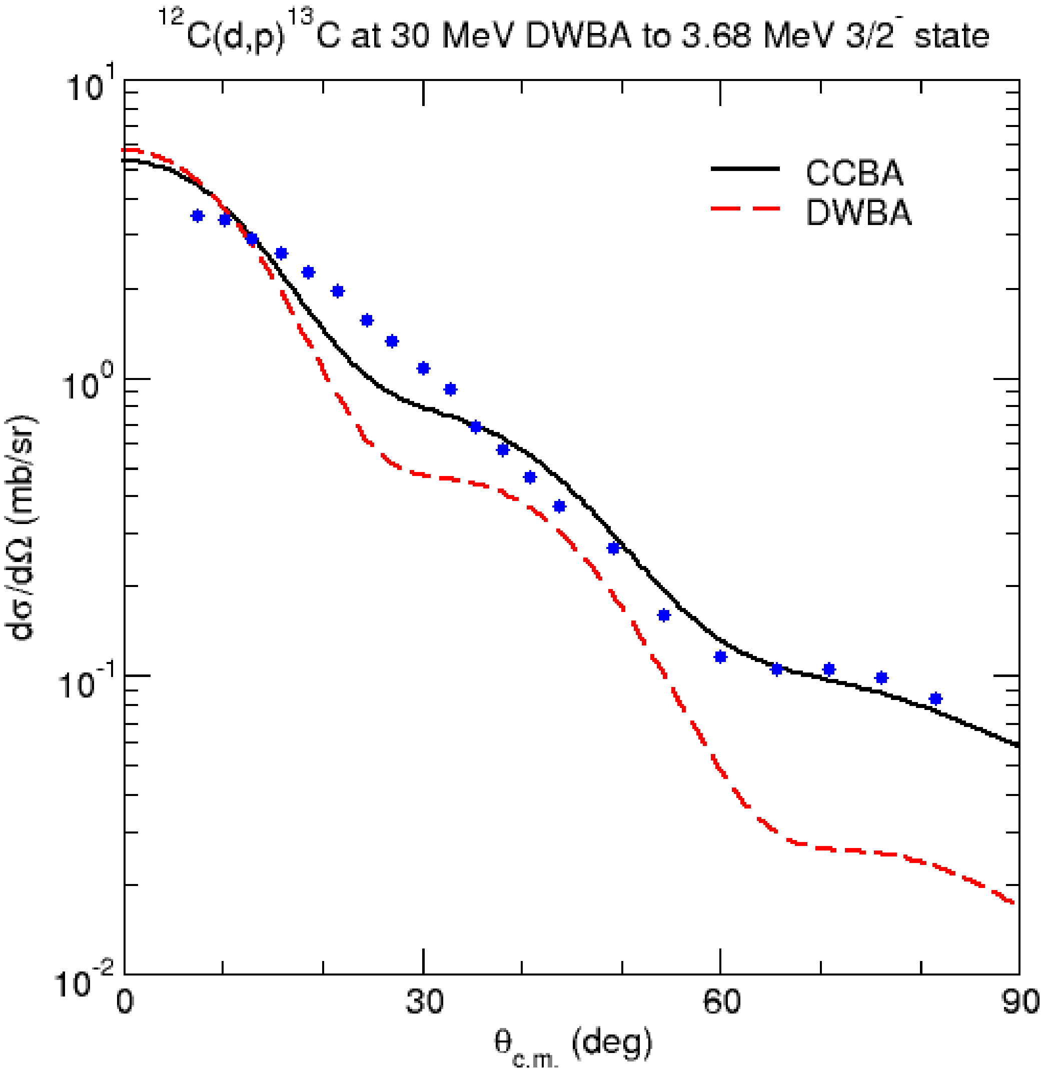
If we do all this, we can now (in principle) search on the 5 SAs until we get the best fit to the data.

This is rather a lot of parameters to fix from one angular distribution (the shade of von Neumann has come back to haunt us!) so we will omit the two $\ell = 3$ components and search on the remaining three SAs:

These are for the following components:

^{12}C $0^+ \otimes 1p_{3/2}$, ^{12}C $2^+ \otimes 1p_{1/2}$ and ^{12}C $2^+ \otimes 1p_{3/2}$

If we do this for the 3.68 MeV ^{13}C $3/2^-$ state what do we find?



SAs are: ^{12}C $0^+ \otimes 1p_{3/2} = 0.373$, ^{12}C $2^+ \otimes 1p_{1/2} = -0.881$, and ^{12}C $2^+ \otimes 1p_{3/2} = 0.454$

We see that the CCBA fit to the data is much better (at the price of an additional two adjustable parameters with all that this implies) and in this case the SA for the ^{12}C $0^+ \otimes 1p_{3/2}$ component has not changed.

However, the reproduction is still far from perfect: the calculated AD is too structured compared to the measured one.

This could be due to the missing $\ell = 3$ components, components built on other excited states of the ^{12}C core (e.g. the 9.63 MeV 3^-) or the effect of deuteron breakup.

This provides a convenient point to consider the question of how the effects of deuteron breakup may be explicitly included in a transfer calculation.

The simplest way is to use the *adiabatic model*.
[Johnson and Soper, Phys. Rev. C **1** (1970) 976, Harvey and Johnson, Phys. Rev. C **3** (1971) 636]

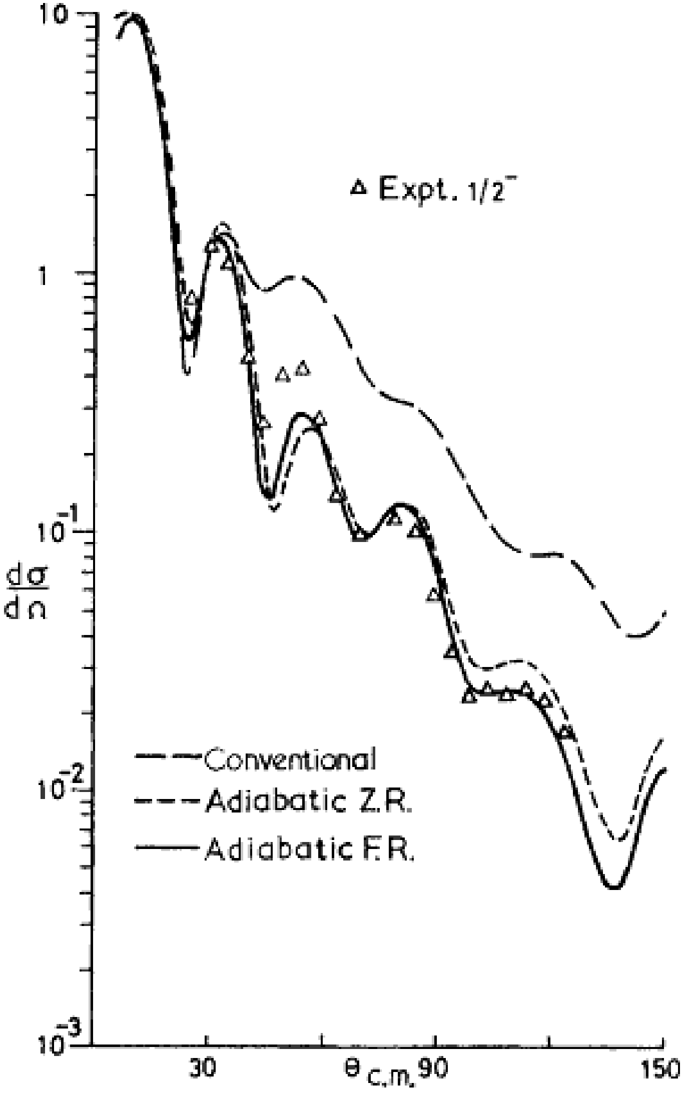
This model keeps the DWBA formalism but the deuteron distorted wave (entrance channel for (d,p) or exit channel for (p,d) reactions) is redefined. It still describes the motion of the centre of mass of the neutron and proton but they may no longer be in the form of a bound deuteron.

This is done by introducing an adiabatic potential into a standard DWBA code in lieu of the usual deuteron OMP. The potential is adiabatic because it ignores the “excitation energy” of the deuteron compared to its incident energy.

We will not discuss how the adiabatic potential is calculated here, but there are approximations available that make it a simple job.

Therefore, the adiabatic model should work best for higher incident energies. In fact, it is found to work well even at fairly low E_d values, due to the errors in the assumptions largely cancelling each other out.

The use of the adiabatic model can lead to significant improvement in the description of experimental data, e.g. ^{54}Fe(d,p)^{55}Fe at 23 MeV:



Note that the adiabatic model in this form will ***not*** describe the deuteron elastic scattering (remember that the “deuteron” distorted wave was redefined), although this is possible within the framework of the adiabatic model theory [Johnson and Soper. Phys. Rev. C **1** (1970) 976]

Taken from Harvey and Johnson, Phys. Rev. C **3** (1971) 636

The disadvantage of the adiabatic model is that it can only be used as a replacement for DWBA; it cannot be adapted to CCBA scenarios. Also, it is not suitable for use within the coupled reaction channels (CRC) formalism, which we shall discuss presently.

A more accurate theory for the modelling of breakup exists, known as the *coupled discretised continuum channels* (CDCC) method, due to Rawitscher, Phys. Rev. C **9**, 2210 (1974). We will not go into details here, but suffice it to say that it is an extension of the standard CC theory to include couplings to the continuum.

CDCC can be used to model the deuteron breakup in a CCBA-type calculation (target excitation can be included) and is also suitable for use in a CRC calculation.

It can also be used in other reactions involving deuterons such as (d,t), (d,^{3}He), (d,^{6}Li) etc., which the adiabatic model can not.

CDCC is therefore more flexible than the adiabatic model but requires much more computing time since the continuum couplings are explicitly included in a CC type calculation (and there are usually many of them).

The final refinement that we can make in our reaction model is to treat the transfer step(s) on the same footing as the other couplings, i.e. no longer treat them as weak so that they can be handled with perturbation theory, as in DWBA.

To do this we use so-called *coupled reaction channels* or CRC. In principle this is CC theory with the extra complication that the exit channel now belongs to a different partition.

In practice it is handled by iterating a series of DWBA steps until the result converges, since “full” CRC would be very computationally intensive (the two methods give identical results).

CRC is available in FRESCO and is usually used in finite range form. In addition to including the “remnant” term we mentioned in the first lecture on transfer reactions there is a correction that must also be included in a CRC calculation, the non-orthogonality correction.

This is a term in the residual interaction that arises in CRC because the exit channel wave functions are not orthogonal to those in the entrance channel (they form a distinct complete set by themselves).

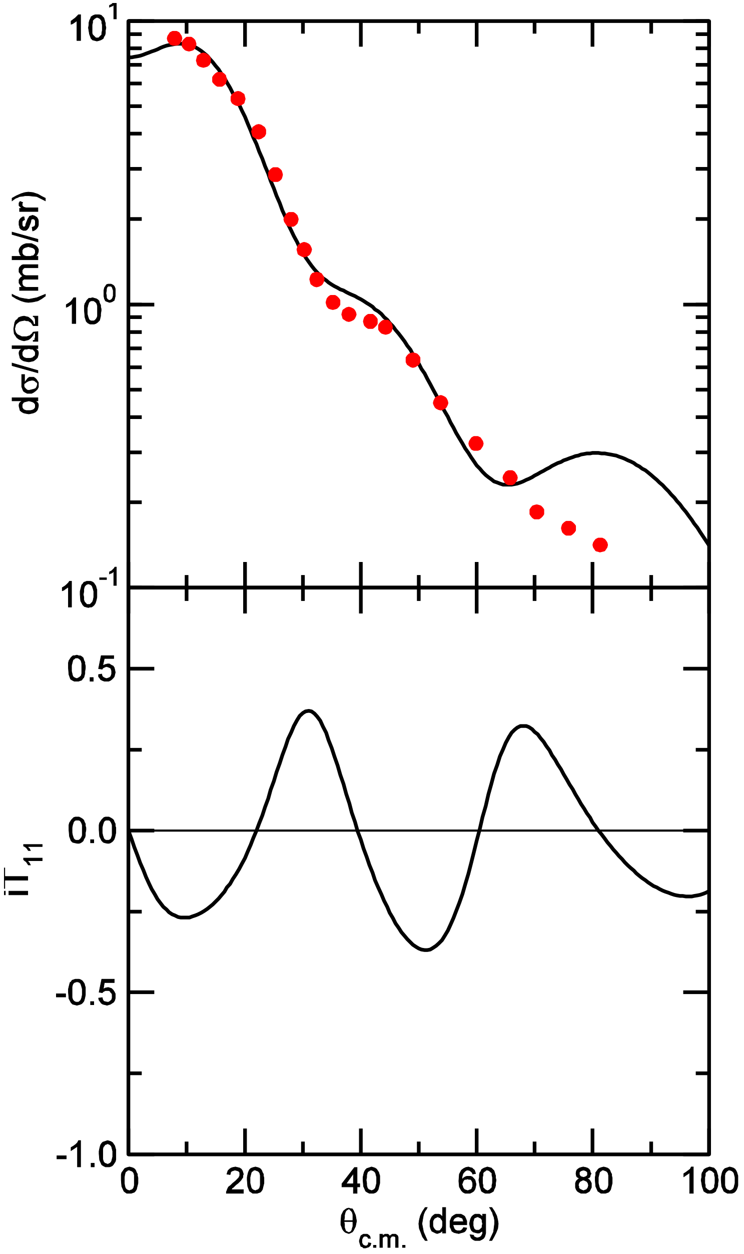
It is not always small and usually cannot be neglected. It can be handled by FRESCO.

Putting it all together: keeping to our ^{12}C(d,p)^{13}C example how well can we describe these data using the most sophisticated model of the reaction mechanism we have available?

This is CC for the inelastic excitation of the ^{12}C 2^+ state, CDCC for deuteron breakup and CRC for transfer steps, including two step via the ^{12}C 2^+ state.

How well does all this apparatus work in reality?

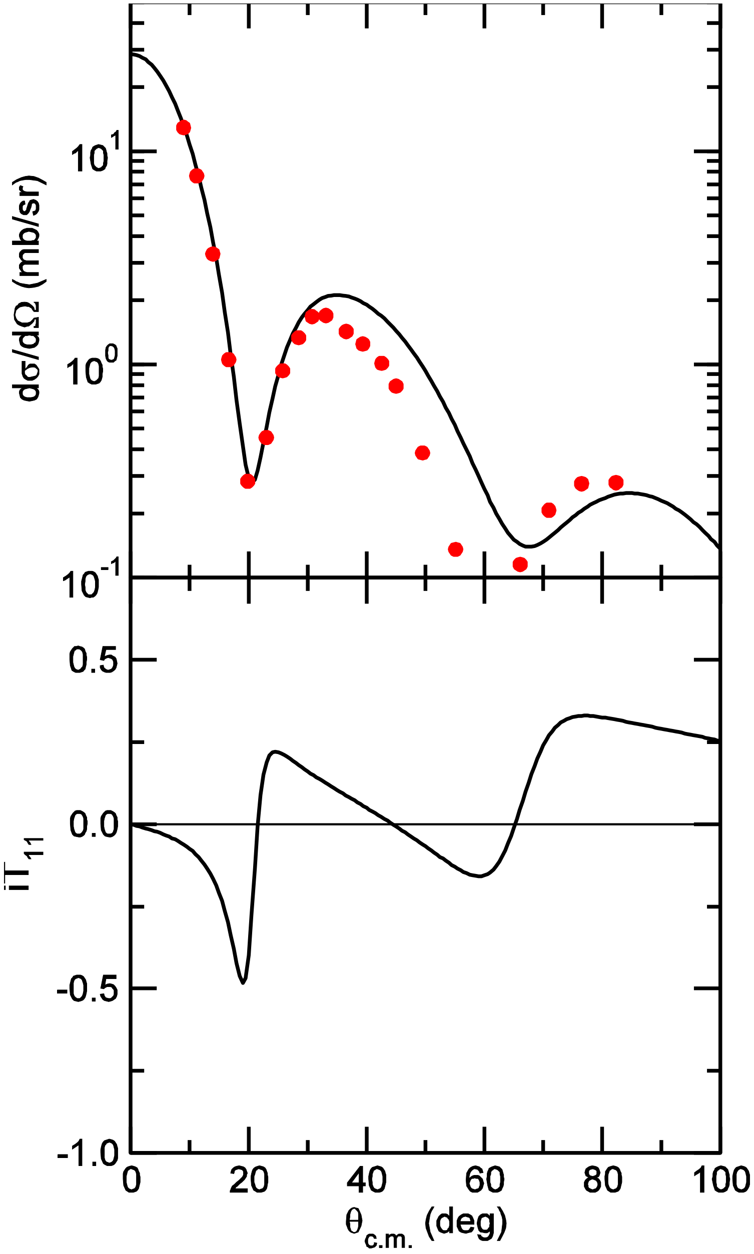
30 MeV $^{12}C(d,p)$ to the 0.0 MeV $1/2^-$ state in ^{13}C



Description is good: shape of AD well reproduced

There are significant effects on the extracted SA too, compared to DWBA or CCBA

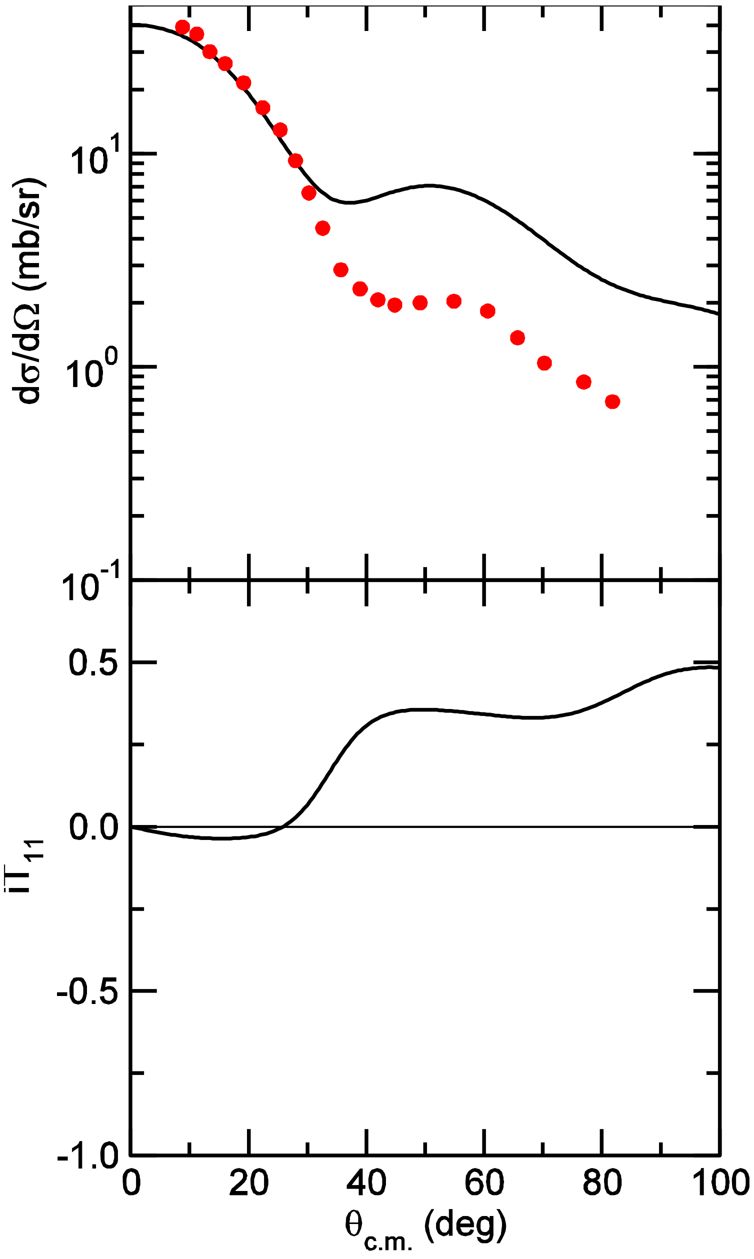
30 MeV $^{12}C(d,p)$ to the 3.09 MeV $1/2^+$ state in ^{13}C



Description is good: shape of AD well reproduced

There are significant effects on the extracted SA too, compared to DWBA or CCBA (mainly due to the fact that only the “full” reaction theory can reproduce accurately the shape of the AD).

30 MeV ^{12}C(d,p) to the 3.85 MeV $5/2^+$ state in ^{13}C



Description is not as good: shape of AD not so well reproduced

There are significant effects on the extracted SA too, compared to DWBA or CCBA

We see that the “full” model of the reaction in general leads to a much improved description of the data (there are still problems with the $5/2^+$ state)

Not only are the shapes of the ADs better reproduced but there are also significant changes in the spectroscopic amplitudes (or factors) that we extract from the normalisation of the theory curves to the data.

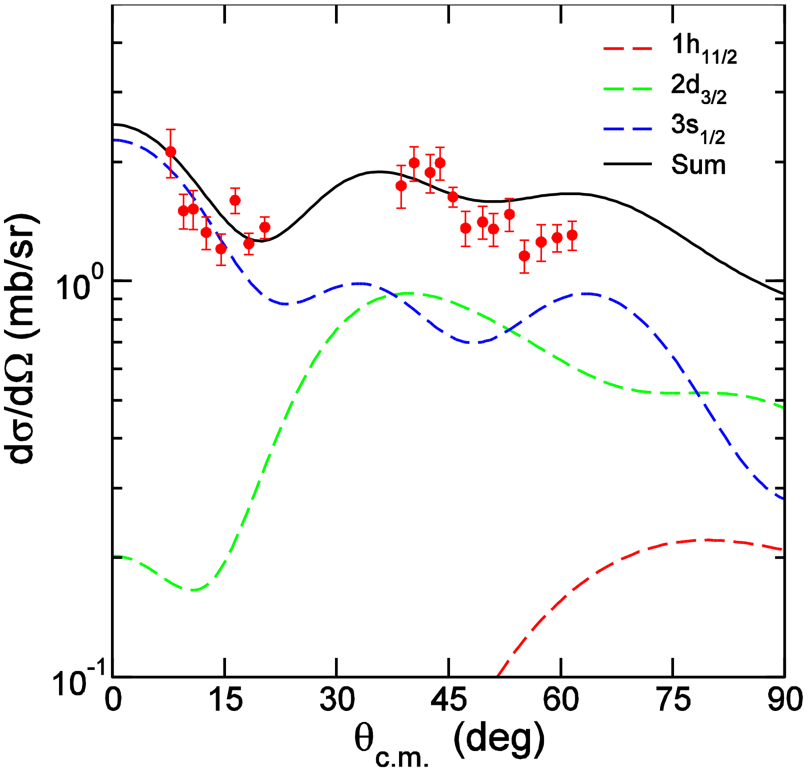
Thus, failure of DWBA to describe transfer data can indicate real physics – it may mean that we need to use a more sophisticated reaction model because the assumptions underlying DWBA are no longer valid

However, to avoid giving you the wrong impression, DWBA works very well when the conditions are fulfilled. As a case study, we take the ^{124}Sn(d,p)^{125}Sn reaction at E_d = 9 MeV.

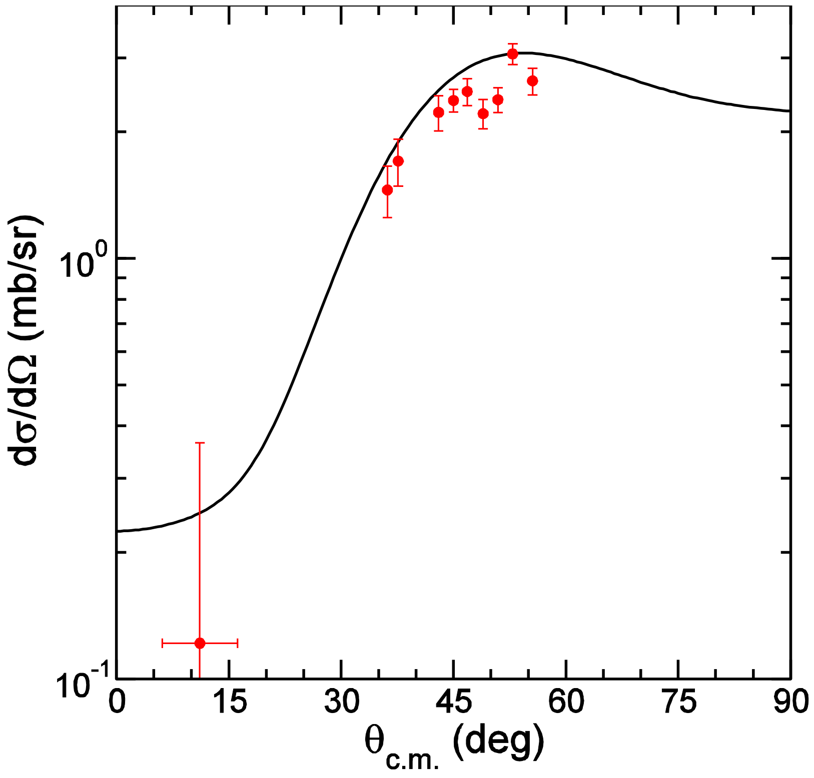
These data also provide a good example of “typical” radioactive beam data (although ^{124}Sn is effectively stable these data were taken in inverse kinematics as a test of a RIB set-up and analysis techniques).

Data from: Jones *et al.*, Phys. Rev. C **70** (2004) 067602

First, DWBA: no elastic scattering data available at suitable energies, so global OMPs were used.

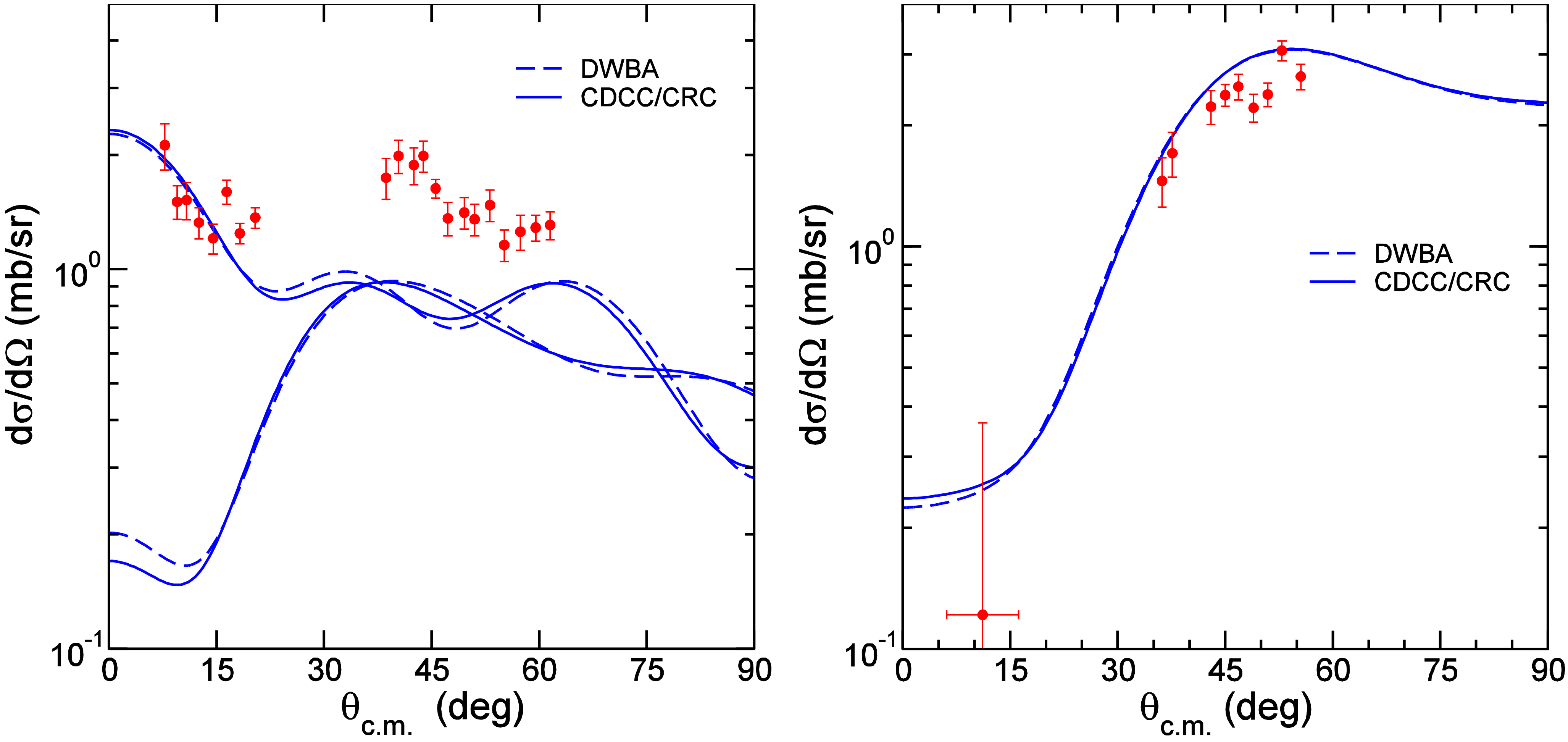


Mixture of 0.0 MeV $11/2^-$, 0.028 MeV $3/2^+$ and 0.215 MeV $1/2^+$ states



2.8 MeV $7/2^-$ state

DWBA describes data well, how does it compare to a CDCC/CRC analysis?



DWBA and CDCC/CRC give essentially identical results in this case, *provided* that the CDCC/CRC calculation reproduces the d + ^{124}Sn elastic scattering predicted by the optical model potential used in the DWBA

Finally, a word about spectroscopic factors/amplitudes

We saw in the first lecture on transfers that the shape of the angular distribution can give us the ℓ value of the final state, an important piece of information (it places limits on the spin of the state).

Another useful piece of information is the spectroscopic factor. As we said earlier, it should always be borne in mind that this is not an observable and is therefore model dependent.

We have seen that the exact value obtained can depend on the reaction model used.

It is also sensitive to the binding potentials used to bind the transferred particle to the core nucleus. It is most sensitive to the radius of this potential, which is not usually fixed with any degree of certainty.

In CCBA and CRC we may also have to determine several spectroscopic amplitudes from a single AD, so the situation is even worse.

However, we can distinguish with certainty between small and large spectroscopic factors, so we can say which states are good single particle levels

As usual in direct reaction work, we must be careful about making detailed comparisons of absolute numbers extracted from experiment using some model with what is calculated by a structure theory: they are not necessarily the same thing!

In the final lecture we consider transfer reactions induced by heavy ions and why they are interesting.

Lecture 12: heavy ion induced transfer reactions

The previous two lectures have concentrated on transfer reactions induced by light ions. We have seen that such reactions are important sources of nuclear structure information due to their selectivity (preferential population of single-particle states).

In this final lecture we shall look at transfer reactions induced by heavy ions. Here the motivation is rather different since we are mainly concerned with the *reaction mechanism* and the effect of such couplings on the elastic scattering.

It is possible to use heavy ion induced transfer reactions for spectroscopy, but this is not often done due to the need to fix the “projectile-like” spectroscopic factor(s)/wave functions as well as the “target-like” (usually the ones of interest).

e.g. ^{208}Pb(^{12}C,^{11}B)^{209}Bi could be used to look at “single proton” levels in ^{209}Bi instead of the (^{3}He,d) reaction but the $<^{12}\mathrm{C}|^{11}\mathrm{B}+p>$ overlap is much less well known than the $<^{3}\mathrm{He}|d+p>$ one.

There can be advantages to using heavy ion projectiles though, due to *Q matching*

Heavy ion induced transfer reactions are observed to have a strong dependence on the *Q* value of the reaction. This *Q-window effect* is due to the requirement that for maximum transfer amplitude the orbits of the incoming and outgoing particles before and after the transfer are continuous.

This means that for a given reaction there is an optimum *Q* value that favours it. For neutron transfers this is always 0 MeV. For proton transfers it is negative for stripping (as in our (^{12}C,^{11}B) example) and positive for pickup (e.g. the (^{12}C,^{13}N) reaction).

There are several ways of calculating Q_{opt}, one of the simplest being:

$$Q_{\mathrm{opt}} = \left[\frac{Z_{\mathrm{b}} Z_{\mathrm{B}}}{Z_{\mathrm{a}} Z_{\mathrm{A}}} - 1\right] E_{\mathrm{cm}}$$

for the reaction A(a,b)B. A more accurate expression, due to Brink is:

$$Q_{\mathrm{opt}} = \left(Z_{\mathrm{b}} Z_{\mathrm{B}} - Z_{\mathrm{a}} Z_{\mathrm{A}}\right) e^2/R - \frac{1}{2} m v^2$$

R is the distance of closest approach, *m* the mass of the transferred particle and *v* the relative velocity of the target and projectile

There is also an optimum angular momentum transfer, formulated by Brink:

$$L_{\mathrm{opt}} = \mid L_{\mathrm{a}} - L_{\mathrm{b}} \mid$$

It is the difference in angular momenta between the incoming and outgoing orbits whose distance of closet approach is *R*.

If we take *R* = strong absorption radius then L_{a} and L_{b} are those angular momenta for which the appropriate elastic scattering amplitudes are given by $|S_{L\mathrm{a}}| \approx |S_{L\mathrm{b}}| \approx 1/\sqrt{2}$

These matching conditions mean that with careful choice of the projectile it is possible preferentially to populate particular levels with heavy ion reactions.

This can be useful for spectroscopic purposes, but as we said previously, it is difficult to extract *absolute* spectroscopic factors from such reactions.

The *relative* C^2S value are fairly well fixed, since the normalisation of calculation to data fixes the product S_aS_A and S_a is the same for all transitions, so the *relative* values should match well with those obtained from light ion reactions.

Turning now to a practical example, we choose the ^{208}Pb(^{12}C,^{11}B)^{209}Bi single proton stripping reaction at an incident energy of 97.9 MeV.

Data for this reaction exist for 5 states in ^{209}Bi:
0.0 MeV $9/2^-$, 0.90 MeV $7/2^-$, 1.61 MeV $13/2^+$, 2.84 MeV $5/2^-$ and 3.12 MeV $3/2^-$. There are also data for the elastic scattering in the entrance *and* exit channels at suitable energies.

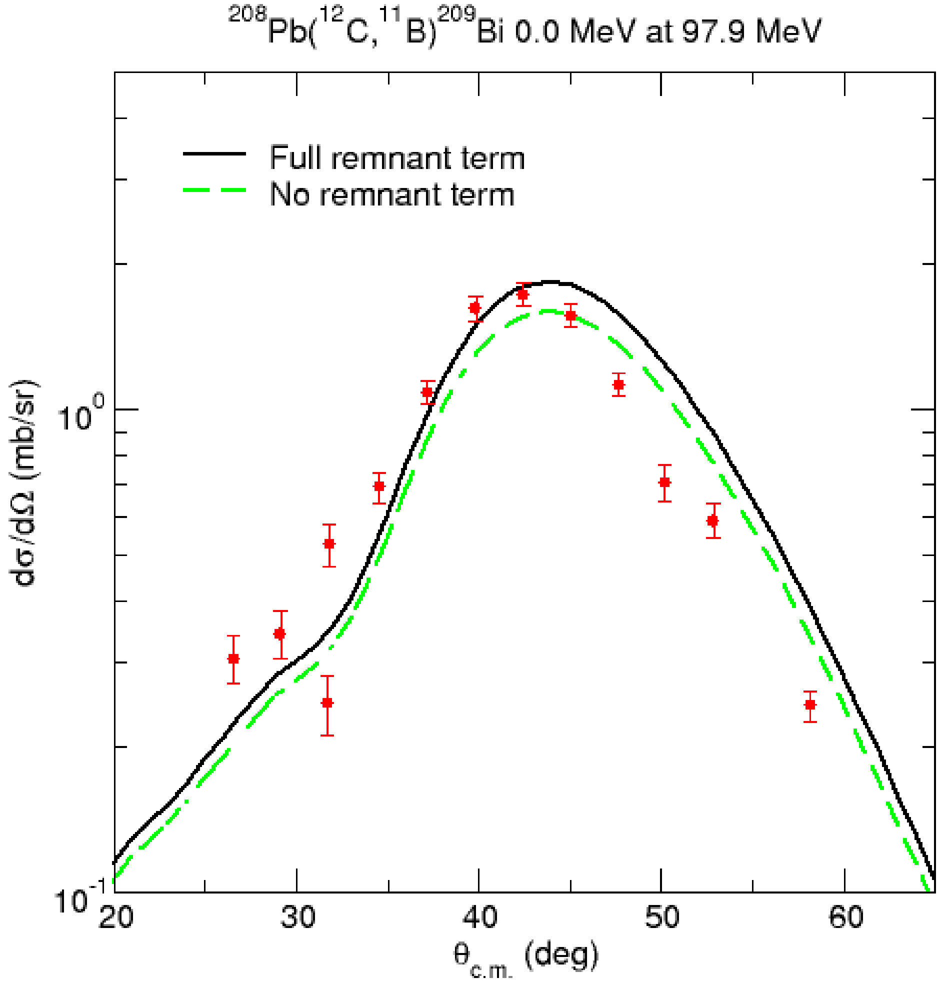
We first repeat the original DWBA analysis in
K. S. Toth *et al.*, Phys. Rev. C **14** 1471 (1976).

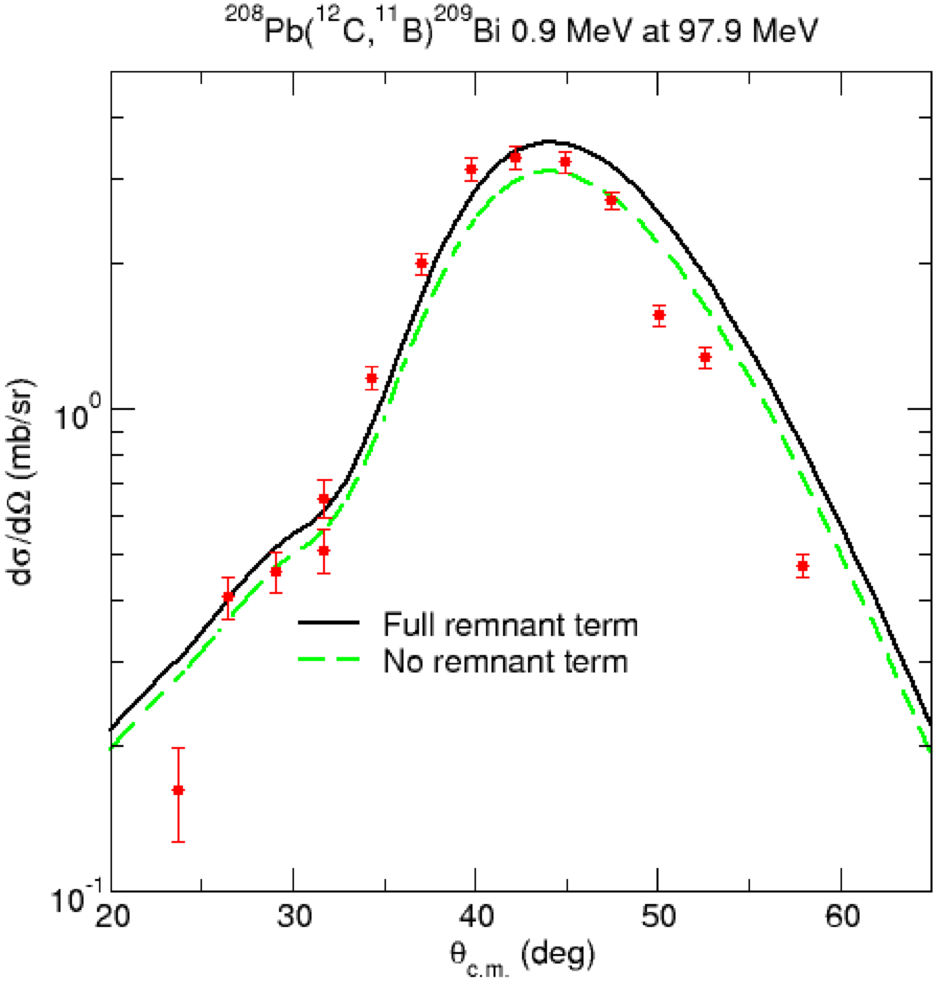
Entrance and exit channel distorted waves are calculated using the OMP parameters given in the original paper. We do the same thing for the bound state wave functions for the projectile and target overlaps.

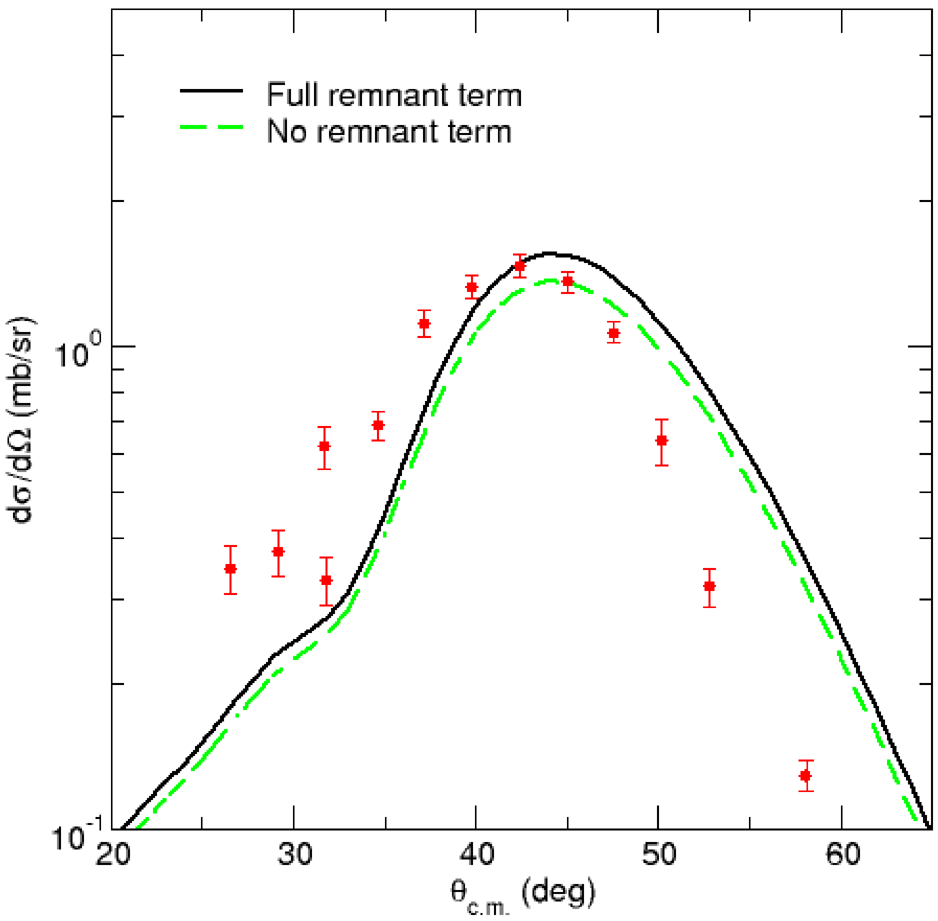
For the projectile overlap the procedure is one that is found to work well for light, p-shell projectiles. We take the calculated spectroscopic factors of Cohen and Kurath, Nucl. Phys. A **101**, 1 (1967) and use a Woods-Saxon well of radius 1.25 x $A^{1/3}$ and diffuseness 0.65 fm.

N.B. here $C^2 \neq 1$ since the isospin Clebsch-Gordan coefficient for $^{12}C = {}^{11}B + p$ is $1/\sqrt{2}$

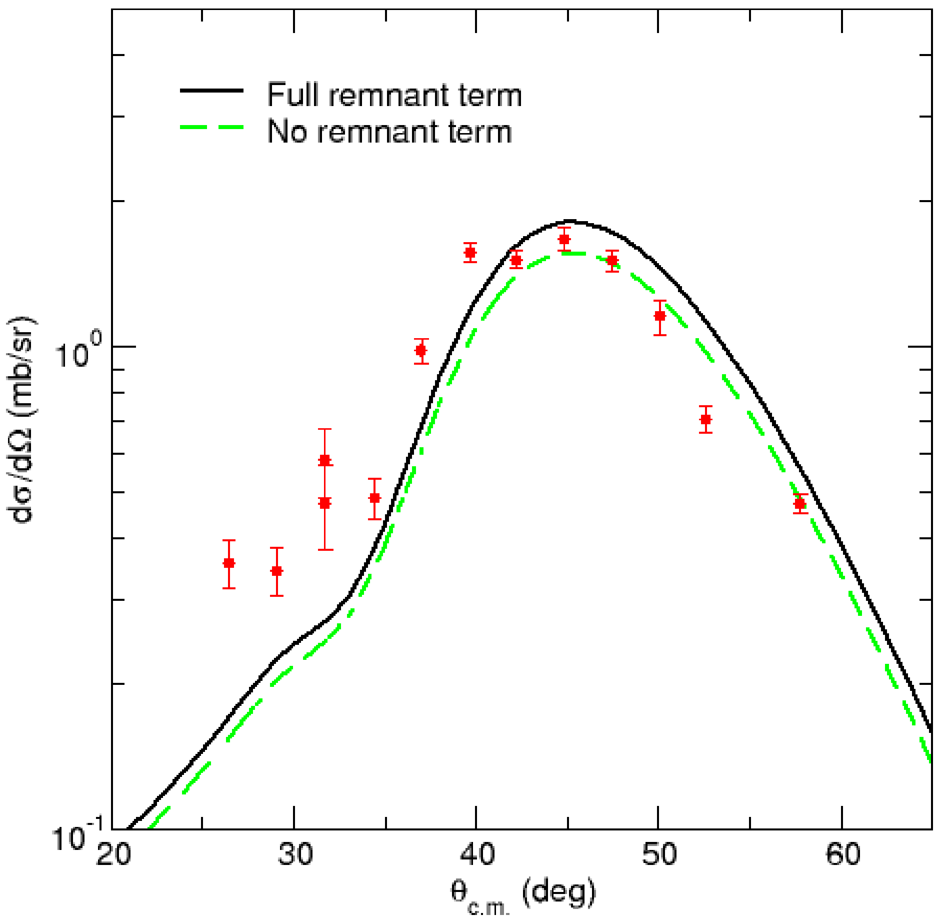
How well does DWBA work in this case? Full finite-range post formulation (zero-range is not appropriate for heavy ions) + remnant term.



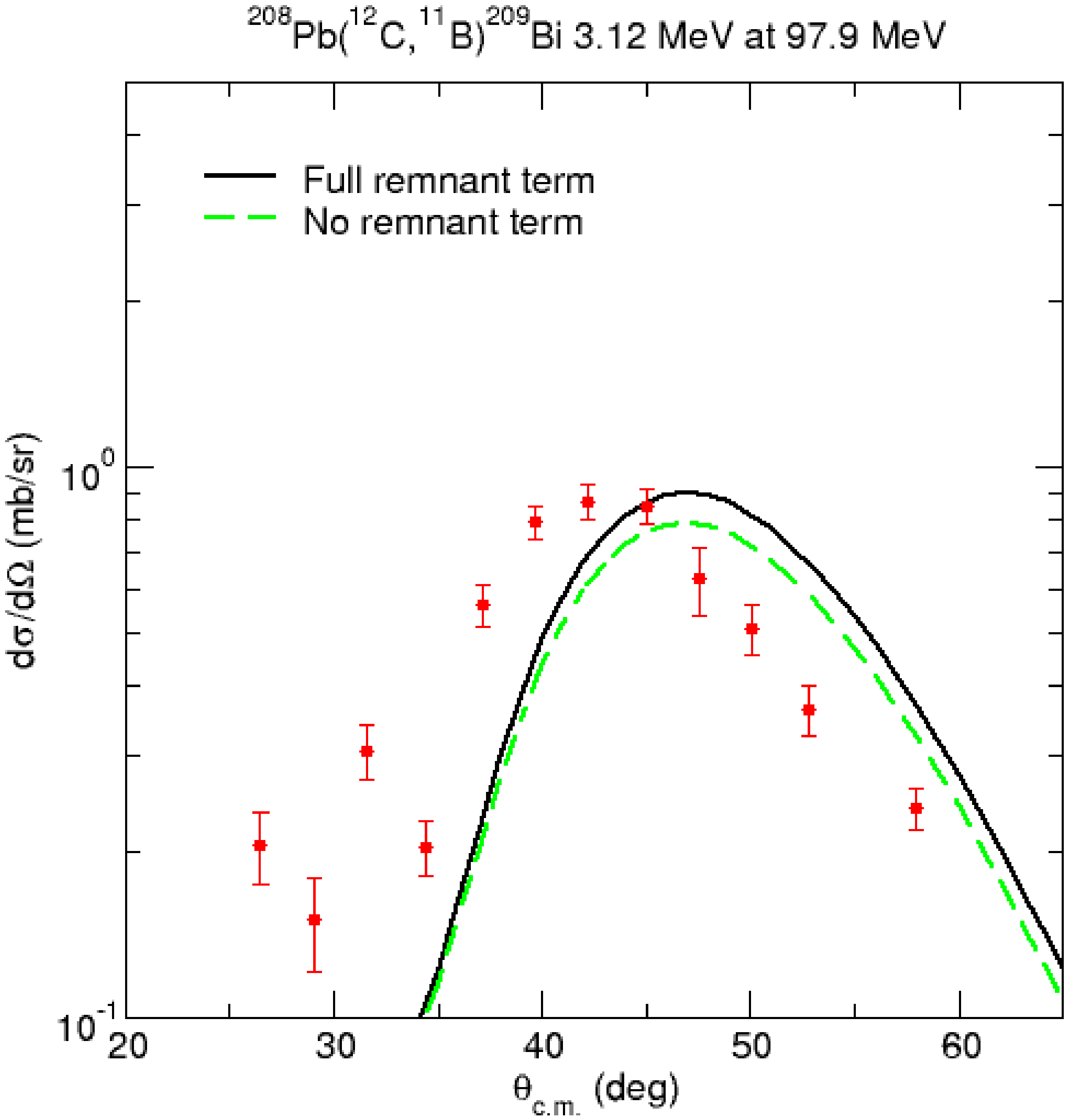




^{208}Pb(^{12}C,^{11}B)^{209}Bi 1.61 MeV at 97.9 MeV



^{208}Pb(^{12}C,^{11}B)^{209}Bi 2.82 MeV at 97.9 MeV



$^{208}\mathrm{Pb}(^{12}\mathrm{C},^{11}\mathrm{B})^{209}\mathrm{Bi}$ 3.12 MeV at 97.9 MeV

The overall description is reasonable, although there are a few things to note:

1) If we use the full remnant term post and prior give identical results

2) Omission of the remnant term makes a significant difference, cf. dashed green and solid black curves

3) There is a systematic shift to larger angles of the calculated transfer peak compared to the measured one (most notable for the 3.12 MeV $3/2^-$ state).

The <^{209}Bi|^{208}Pb+p> spectroscopic factors turn out to be quite reasonable when compared to other measurements and theoretical values, although the latter comparison is always subjective.

However, the discrepancy in the peak positions suggests some problems with simple DWBA.

It is possible that choice of OMP affects the result, and choice of bound state potential radius may do so too. These possibilities were checked and had little influence on the shape of the transfer ADs.

This suggests that the reaction mechanism may be more complicated than the simple one-step direct assumed in DWBA.

Two-step mechanisms – inelastic excitation of states followed by transfer – can be more important for heavy ions, since in addition to excitation of the target-like nuclei it is now possible to excite the projectile-like nuclei too.

In the ^{208}Pb(^{12}C,^{11}B)^{209}Bi reaction there are possible collective excitations of the ^{208}Pb core (we expect excitations of ^{209}Bi to be weaker) plus the ^{12}C projectile and ^{11}B ejectile (these should also be weaker)

We consider the two-step process of excitation of the ^{12}C 4.44 MeV 2^+ state followed by stripping to the ground state of ^{11}B. This should be the most important such process, since the ^{12}C 2^+ is the most strongly coupled excited state.

The C-K spectroscopic factor for the $<^{12}C(2^+)|^{11}B+p>$ overlap is reasonably large, so the effect of this new reaction path may be significant.

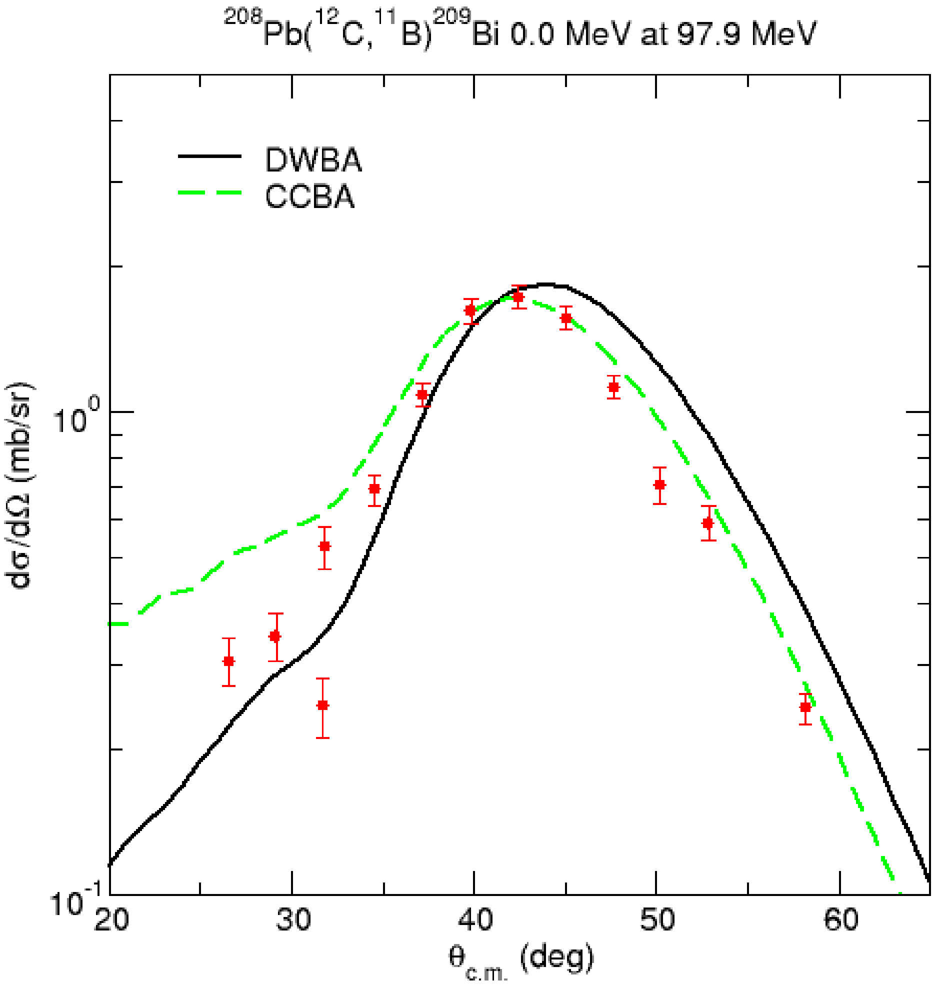
We include inelastic excitation of the ^{12}C 4.44 MeV 2^+ state in the usual way using the CC formalism and retain DWBA for the transfer steps (CCBA).

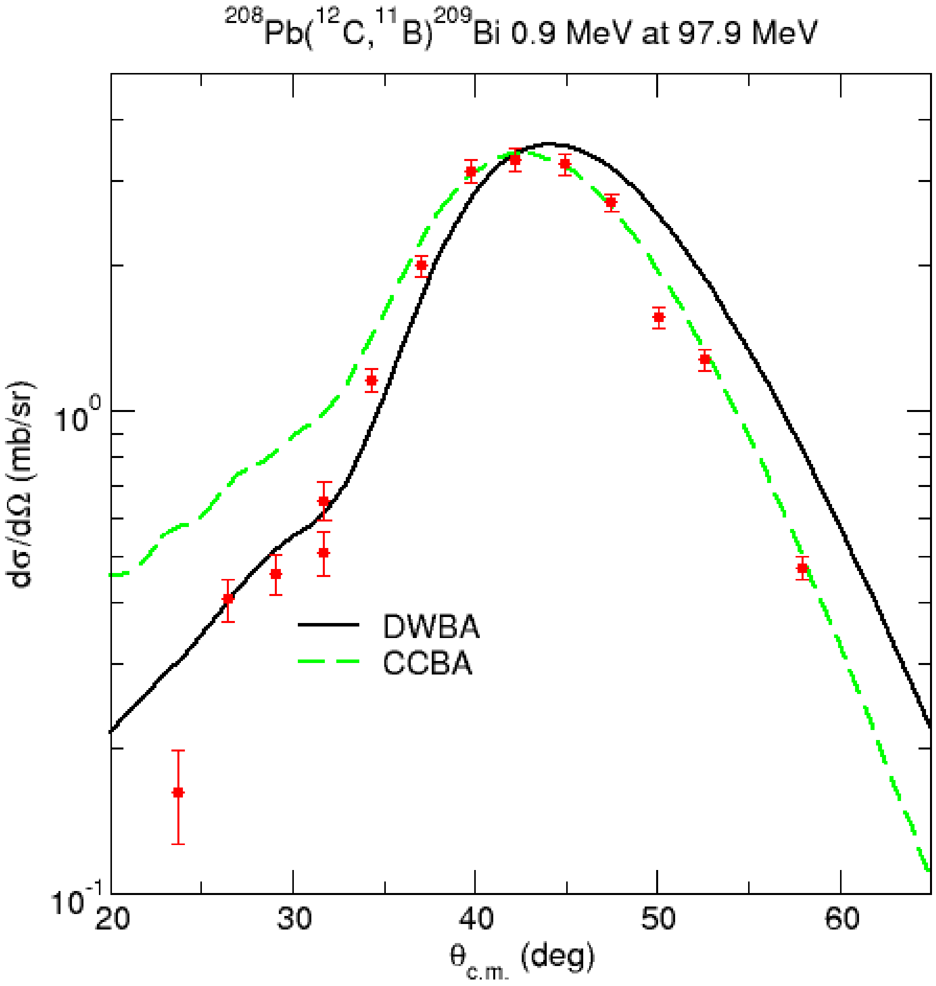
We also take care to readjust the OMP in the entrance channel to recover the same elastic scattering AD from the CC calculation as we had from the original OM one.

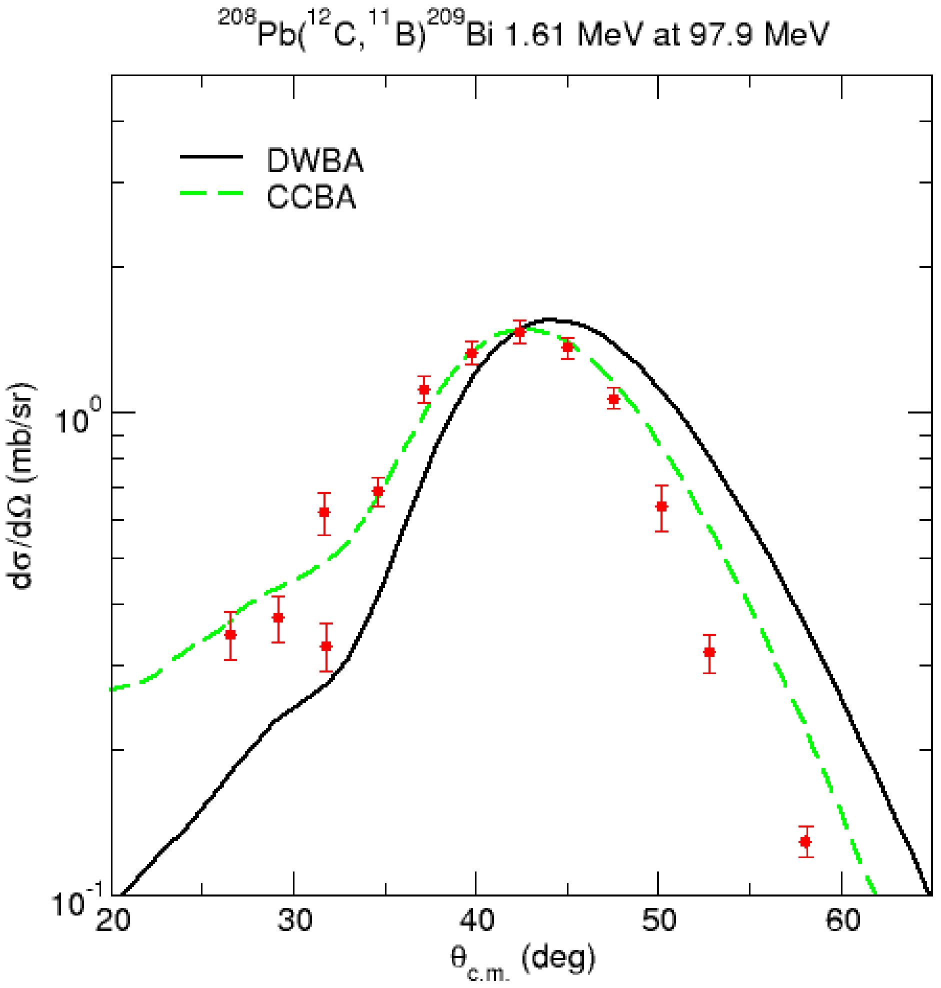
Recall that in CCBA we need the spectroscopic *amplitudes*. C-K only give spectroscopic factors, so we do not *a priori* know the relative sign of the $<^{12}C(0^+)|^{11}B+p>$ and $<^{12}C(2^+)|^{11}B+p>$ SAs.

We therefore treat the sign as a parameter and try both +ve and −ve to see which gives the best description of the transfer data.

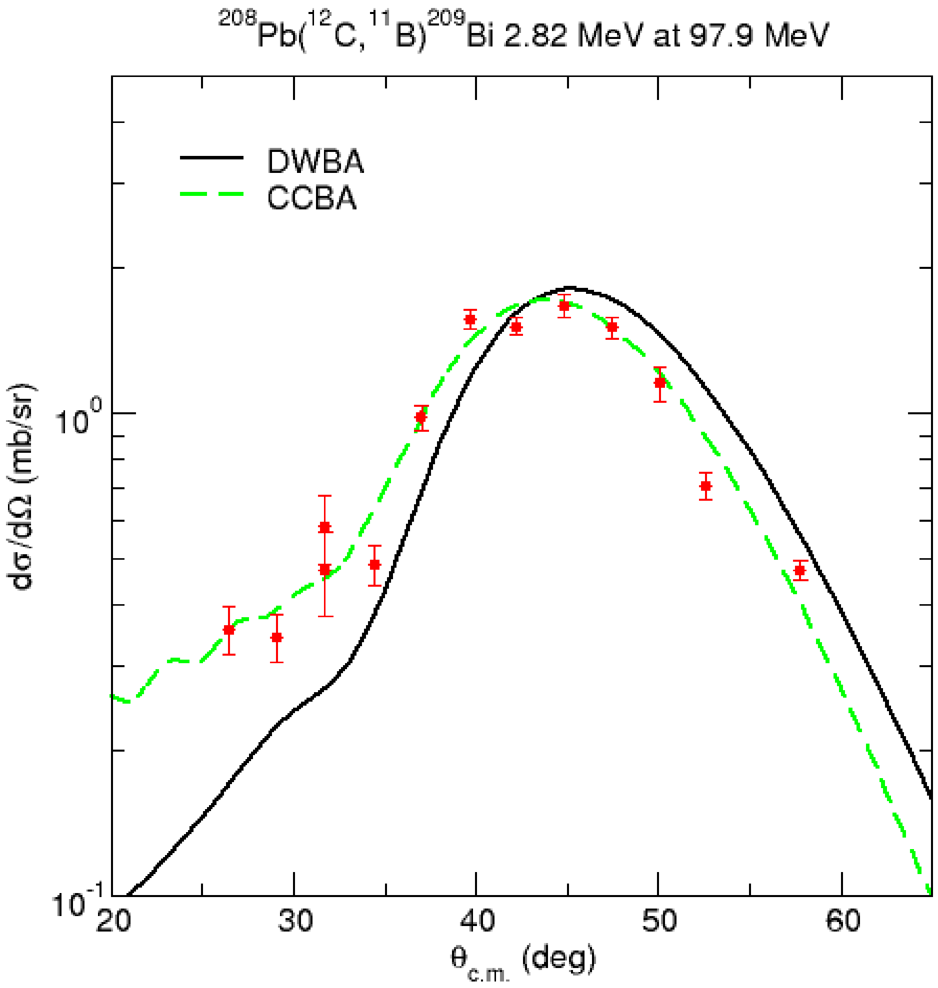
How well does CCBA describe the ^{208}Pb(^{12}C,^{11}B) data?





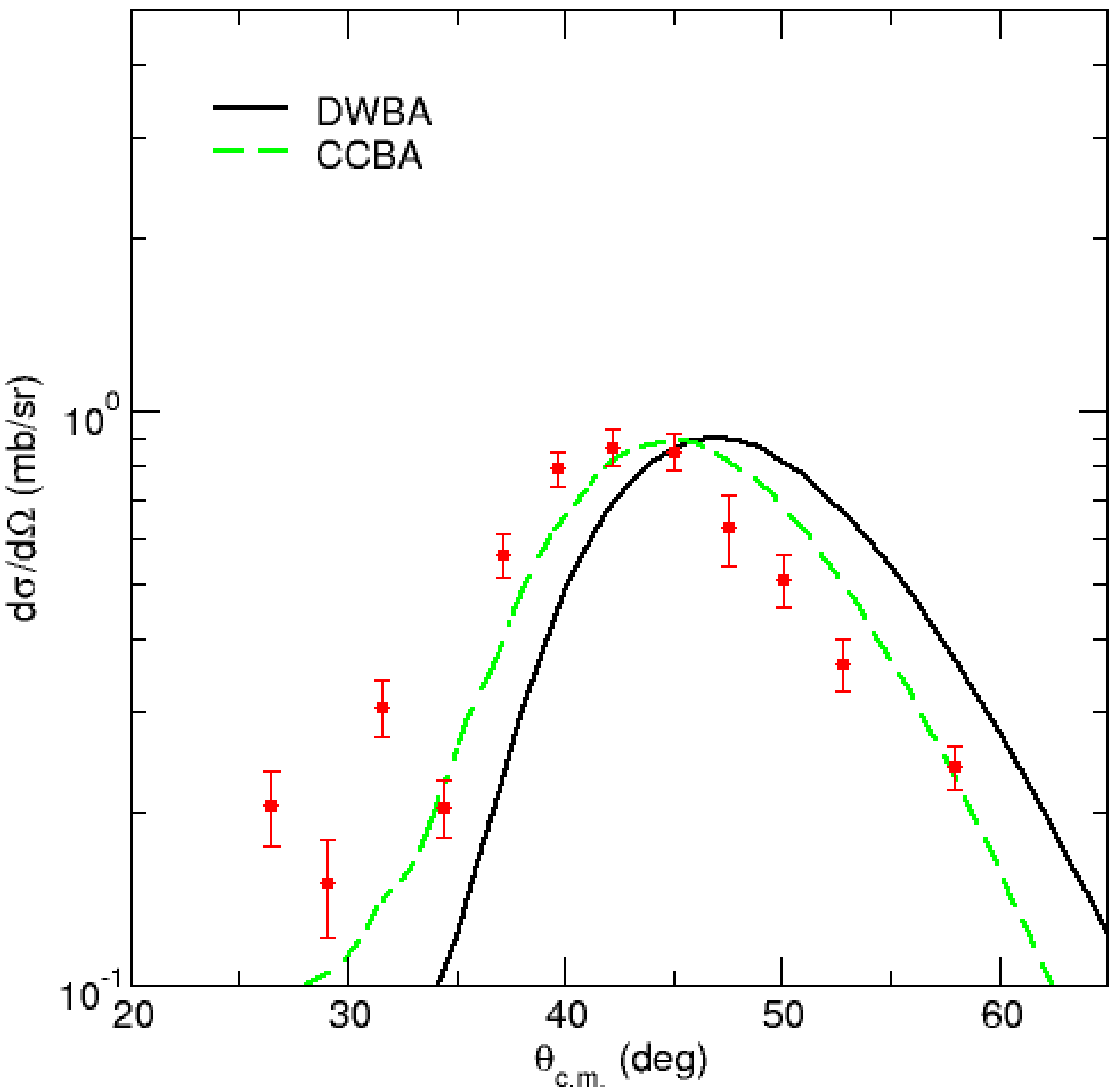


^{208}Pb(^{12}C, ^{11}B)^{209}Bi 1.61 MeV at 97.9 MeV



^{208}Pb(^{12}C, ^{11}B)^{209}Bi 2.82 MeV at 97.9 MeV

^{208}Pb(^{12}C,^{11}B)^{209}Bi 3.12 MeV at 97.9 MeV



CCBA is able to provide a very good description of all the data, with the exception of stripping to the 3.12 MeV $3/2^-$ state where there is still a mis-match between the position of the measured peak in the AD and the calculated one (but it is much improved compared to the DWBA result).

The reaction mechanism does indeed seem to be more complicated than single-step (and perhaps even three-step or more paths contribute to stripping to the 3.12 MeV $3/2^-$ state).

There are two things to note from this result:

1) Heavy ion transfer reactions can be sensitive to details of the reaction mechanism

2) The relative sign of the $<^{12}C(0^+)|^{11}B+p>$ and $<^{12}C(2^+)|^{11}B+p>$ spectroscopic amplitudes is unambiguously fixed by this analysis; it must be negative otherwise the peak position shifts to larger angles, i.e. in the wrong direction

We do gain some useful spectroscopic information from this heavy ion transfer after all, in addition to the interest of pinning down the reaction mechanism.

Another motivation for studying heavy ion transfer reactions is their possible influence on the elastic scattering. For these studies we must use the CRC theory, since in both DWBA and CCBA the transfer step is treated as a perturbation and so cannot influence the elastic scattering.

We will take as an example some new data for a radioactive beam, ^{8}He, incident on a ^{208}Pb target at 22 MeV, close to the Coulomb barrier.

Energies close to the Coulomb barrier are where coupling effects (of any sort) are most important for the elastic scattering.

^{8}He should prove an interesting case. The *Q* matching window should favour 1n-stripping to low-lying levels of ^{209}Pb, which are known to be good single-particle levels.

Also, the $<^{8}\mathrm{He}|^{7}\mathrm{He}+n>$ and $<^{209}\mathrm{Pb}|^{208}\mathrm{Pb}+n>$ spectroscopic factors are well known, so a calculation of the ^{208}Pb(^{8}He,^{7}He)^{208}Pb stripping reaction should give a realistic result.

This is an important point since we cannot check the accuracy of the calculated transfer angular distributions in this case since ^{7}He is unbound (and coincidence measurements with neutrons are difficult)

There are at least two schools of thought as to how to go about a CRC study of transfer coupling effects on the elastic scattering:

1) Take a “bare” OMP consisting of a double-folded real part and an imaginary part that simulates absorption due to compound nuclear processes (fusion) only. Then build up all other sources of absorption by coupling explicitly with CC (inelastic) and CRC (transfer) formalisms to all the important channels.

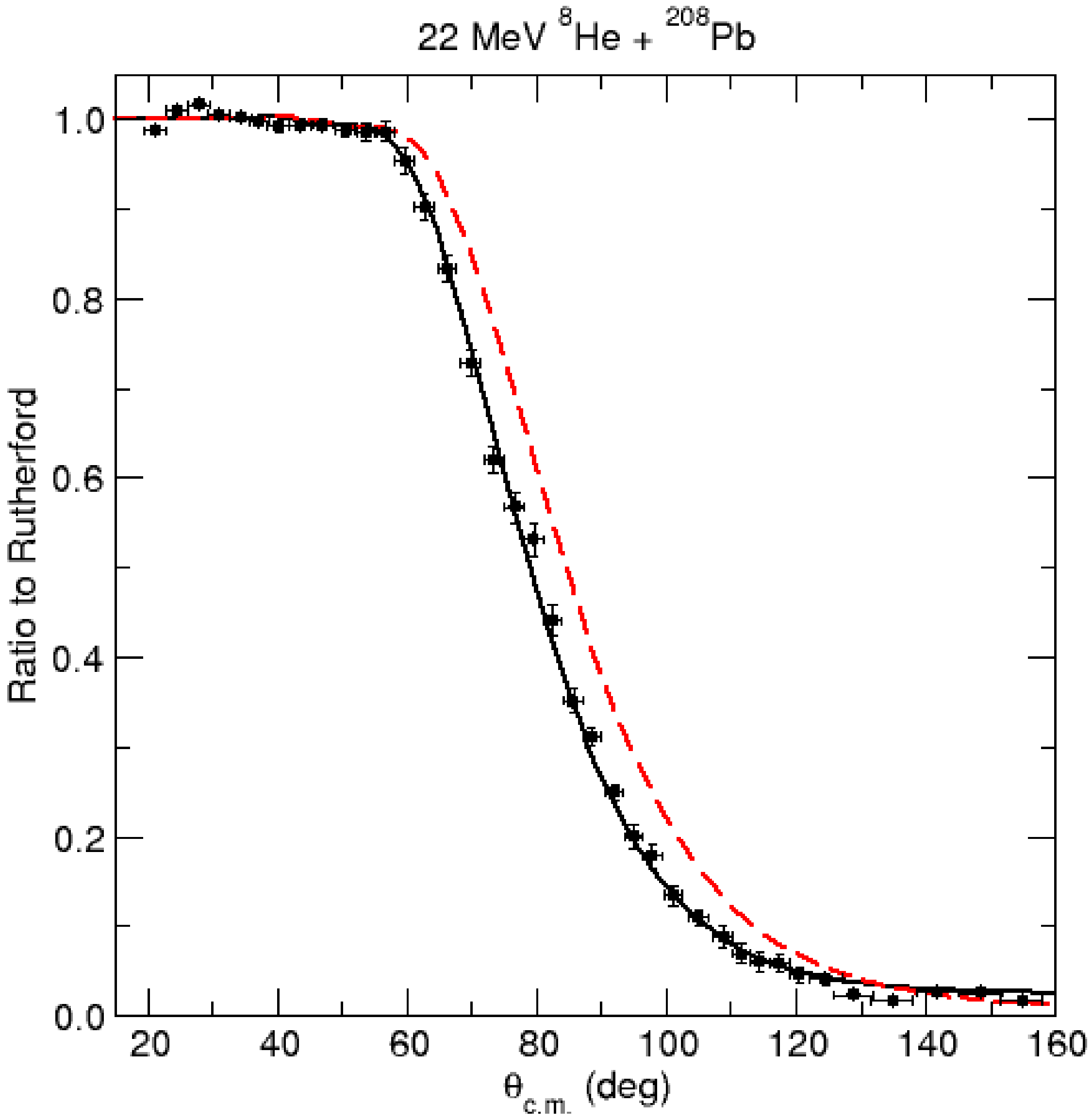
2) Start from the OMP that fits the elastic scattering and readjust its parameters each time you add a coupling to recover the fit.

Both approaches have their merits, and in principle if one could always include all the important channels in the calculation they would eventually give the same results.

In our test case, ^{8}He + ^{208}Pb, we will use the second approach since we are interested in the effect of coupling to a single process – 1n-stripping – only. The effect of coupling to other channels is absorbed into the entrance channel OMP.

What does the full CRC calculation look like, and what effect does switching off the 1n-stripping couplings have on the elastic scattering?

The description is good, and the effect of the transfer couplings is strong:



In fact, the 1n-stripping in this case accounts for about 365 mb out of a total reaction cross section of 1517 mb, so about 24 %.

Thus it seems reasonable that the coupling effect on the elastic scattering is large (however, it is not always so, and a large cross section does not guarantee a large coupling effect).

In fact, the 1n-stripping coupling accounts for about 38 % of the total imaginary potential strength as well as making a significant contribution to the real part.

Thus we see that heavy ion induced transfer reactions can also be interesting from the point of view of their influence on the elastic scattering.

This is one of the major motivations for the study of heavy ion reactions in general – their coupling effects on elastic scattering can be important and are often quite sensitive probes of the properties of the colliding nuclei.

We have finally come to the end of this series of lectures on direct reactions and their analysis. I hope that I have given a flavour of the interest and motivation for such studies.

I have not touched on the interesting area of breakup reactions and their analysis, mainly through lack of time.

There are some other details of reaction analysis techniques that I have not mentioned, but you should now have enough knowledge to understand a direct reaction paper and assess it critically.